Gravitational radiation from Schwarzschild black holes: the second order perturbation formalism

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The perturbation theory of black holes has been useful recently for providing estimates of gravitational radiation from black hole collisions. Second order perturbation theory, relatively undeveloped until recently, has proved to be important both for providing refined estimates and for indicating the range of validity of perturbation theory. Here we review the second order formalism for perturbations of Schwarzschild spacetimes. The emphasis is on practical methods for carrying out second order computations of outgoing radiation. General issues are illustrated throughout with examples from "close-limit" results, perturbation calculations in which black holes start from small separation.

I. INTRODUCTION

A common occurrence while studying a physical system is that the situation that one is most interested in is too complicated to treat in closed form. Yet, for the very same system, it might be possible to study in a closed form a situation that has a higher degree of symmetry. An intermediate avenue has usually been to consider the system in a regime that is more akin to the one of interest but that departs "slightly" from the situation of high symmetry for which one can solve exactly. One is therefore studying "small perturbations" of an exact solution. Considering the perturbations as small allows them to be described by a linear theory. The resulting equations usually simplify so much that they can be solved in situations without any particular symmetry, provided they depart only slightly from an exactly known configuration.

This approach is applicable in general relativity. The field equations of the theory are a set of non-linear partial differential equations that are not solvable in many situations of interest. In fact, state of the art computers might not be able to evolve solutions far enough in time for the physics of interest to be captured. Yet, if one assumes spherical symmetry, an exact solution has been known since 1916. The exact solution is the Schwarzschild solution that describes the geometry of a black hole. As the concept of black hole came to better physical understanding in the 1950's, people started studying spacetimes that represented "small departures" from a black hole geometry. Regge and Wheeler [1] were the first to study these departures, and were able to find a clean formulation for one-half of the degrees of freedom of the problem. The formulation for the other half had to wait until the work of Zerilli [2]. Perturbations of flat spacetime had been considered earlier on [3], and in fact Einstein himself discovered an approximate version of the Schwarzschild solution as a perturbation of flat space.

The motivation of Regge and Wheeler to study perturbations of black holes was to assess the stability of the black hole solutions. Were the black hole solutions "an accident" that arose as a consequence of the assumption of spherical symmetry only, or did they persist if perturbed? The studies of Regge and Wheeler and Zerilli, and later on of Price [4] and others indeed showed that the black hole solutions are stable under small perturbations. Stability was also the motivation of the early second-order perturbations studies due to Tomita and Tajima [5]. They were the first to workout the second-order perturbation analysis of the Schwarschild solution with the aim of probing the non-linear stability of the horizon.

Since the earliest studies, perturbation techniques have been useful in probing issues of astrophysical significance in situations without symmetries, that would otherwise be prohibitively complicated to analyze. Examples of these were the analyses of the motion of particles in black hole backgrounds. The formalism was laid out by Zerilli [6] and the first studies done by Davis et al. [7]. These have subsequently been extended to rotating holes and spinning particles (see [8] and references within).

The perturbations of the "exterior" Schwarzschild metric could be coupled to perturbations of interior solutions, and therefore one could analyze perturbations of stellar objects. This is a subject that lies outside the scope of this review. For further references see the work of Cunningham et al. [9] and for more recent references see [10]. Also outside the scope of this review will be the treatment of perturbations of rotating black holes, first treated by Teukolsky [11], which very recently has been extended to second order perturbations [12].

The increased activity in numerical relativity applied to black hole collisions [13], and the possibility of black hole collisions as a source of detectable gravitational waves [14], has renewed interest in calculations based on perturbation theory. Of particular interest have been the "close limit" calculations [15]– [16]. In this technique, the colliding holes start out well inside a single horizon. The initial data, and the spacetime that evolves, outside the horizon, are considered to be perturbations of a final stationary hole formed in the collision. Much of this recent work has been useful in providing checks of numerical relativity results, providing insight into those results, and in cautiously extending some results. The work, so far, has mostly focused on perturbations of Schwarzschild spacetimes, although work has very recently started on perturbations of Kerr holes [17].

An inherent feature of linearized perturbation theory, for this or any application, is that there is no "built in" indication of how good the perturbation approximation is. For close limit work, linearized perturbation theory applies, in principle, only in the limit of zero separation. But in applying this technique, one is not interested so much in points of principle as in practical results. One wants to know that a perturbation calculation should agree with "perfect" numerical relativity results to, say, within 5%. There are rough indicators of when close limit perturbation theory should be applicable. For example, if an apparent horizon surrounds two colliding holes one guesses that close limit evolution of the initial data will give tolerable accuracy. In other contexts, there are often similar indicators of when perturbation theory applies. But these are rough indicators only, and are typically not quantitative. Short of numerical relativity, the only systematic approach to quantifying the errors in linearized theory is the study of higher order perturbations. If one computes a physical quantity correctly to second order in a perturbation parameter, then the difference between that result, and the result of first order theory alone, is a systematic, quantitative, indication of the error in the perturbation theory calculation.

The need for a good error indicator is paramount in situations where one is "pushing the envelope" of perturbation theory, i.e. venturing into large values of the expansion parameter. The case of the "close limit" of black hole collisions is one of the first such situations. One is not really interested in the "close limit" of the collision, one is forced into it since it is tractable, one is really interested in going as far away as possible from that limit. For that reason a significant effort has recently gone into the development of second order perturbations of the Schwarzschild spacetime as a practical tool for calculations. We have reported results of second order calculations for head-on collisions of equal mass holes [18,19], for initially boosted holes [20], and for a single slowly spinning hole as it relaxes to its Kerr final state [16]. Those papers emphasized the results and (especially) the comparison with numerical relativity results. In the present paper we supply a more detailed description of how second order perturbation theory is carried out. A presentation like this one is important if other researchers are to take advantage of the developed tools in other contexts.

As we mentioned before, there has been some work done in the past on second order perturbations. Tomita and Tajima [5] studied second order perturbations in a null formulation, but their interests were in studying the stability of the horizon, and therefore their formalism would have to be significantly reworked to address the issue we are focusing on, outgoing radiation. Cunningham, *et al.* [9] studied the second order perturbations due to the rotation of a star, if one viewed the rotation as a perturbation. Because of the narrower range of applicability of these formalisms we will not cover them in this review.

We will, specifically, describe here a scheme for carrying out second order perturbation calculations of the evolution of an initial value solution. We will assume that on a $t = 0$ hypersurface we have a solution of Einstein's initial value equations for a 3-metric γ_{ij} and an extrinsic curvature K_{ij}, and that these can be expanded to second order in a perturbation parameter. We will assume, then, that we have first and second order perturbative initial data. How are these data to be evolved forward in time and, in particular, how is outgoing radiation to be computed?

We will start with a reasonably careful and complete description of first order perturbation calculations. This may seem unnecessary in view of the venerable status of first order work. But we will pattern almost all our second order equations on their first order equivalents, and it is important that we establish the notation and "style" of the first order approach. This will enormously simplify understanding second order calculations since it turns out that almost all second order equations are obvious translations of first order equations except for the very important addition of "source" terms quadratic in first order perturbations. Patterning the second order equations so closely on the first will allow us, when we introduce second order theory in Sec. IV, to focus, not on complexity, but on gauge issues that are somewhat subtle, and that have no first order equivalent. These gauge issues are potentially confusing but, as will be made clear in Sec. IV, computations of outgoing radiation cannot be carried out without facing these issues.

The presentation of the first order formalism will be rather general. Prescriptions are given for computations starting from any perturbative initial value solution of Einstein's equations, and ending with the outgoing radiation amplitudes. We do not give similarly complete and general prescriptions for the second order calculations. The reason is the complexity of the source terms that appear in the second order equations. For a second order perturbation of a particular multipole index (ℓ, m), and a particular parity (even, odd), there will in general be contributions to source terms from first order perturbations of all multipole indices and from both parities. If we were to give general source terms, the meaning of the calculations would be obscured by the complexity of the expressions, and

the important, but easily overlooked, conceptual issues might be missed. There is a further reason to avoid general source term expressions. Such source terms would consist of infinite series of products of first order terms along with a set of coefficients in the series. In principle, a notation system could be introduced to represent the Clebsch-Gordon-like factors in these coefficients that arise from the projection of a tensor multipole component from the product of two tensor multipoles. In practice, such formalism may be unrelated to the practical way in which source terms are computed. In examples in which many first order multipoles contribute to each second order multipole, it seems plausible that direct numerical methods, rather than formal methods would be used for the computation of the source terms. To be concrete, for most second-order applications, the researcher involved will have to construct the appropriate source terms. Listing pages and pages of expressions in this paper will not be a viable route to tackle the problem, since it is impractical to consider all possible situations. The attempt of this paper is to lay out the general formalism of how to deal with higher order perturbations and in particular to address the issues of gauge fixing and of meaningful extraction of gravitational waveforms. The lessons developed here are general and useful in all cases. The particular expressions will have to be re-worked.

Clarity, of course, will require examples of explicit expressions. For that reason we will take the following approach. Our second order presentation will be restricted to the $\ell = 2$, even parity, vacuum perturbations. Generalization to other second order multipoles is straightforward, and the equivalent analysis for odd parity is significantly simpler. Expressions will not be given for general source terms (source terms for a general system of first order perturbations) but will be illustrated with examples from close limit calculations. It turns out that for the initial data sets that have been used in close limit work, and until recently as a starting point for most numerical work, the only first order perturbations are the $\ell = 2$ even parity perturbations. The Misner initial value solution [21] is a particularly simple example of this. In addition to being important, these cases have anomalously simple source terms. This special class of examples will be used in Sec. IV to give concrete illustrations of source terms, and more generally of nonlinear terms. The reader is cautioned not to mistake these nonlinear expressions as generally valid, .

The remainder of the paper is organized as follows. Sec. IIA starts with an overview of perturbation theory in general including questions of reparameterization and gauge transformations at different orders. The specialization is made in Sec. IIB to perturbations of the Schwarzschild spacetime, and the Regge–Wheeler (RW) notation (not to be confused with the Regge–Wheeler gauge) is introduced [1]. The RW gauge [1] is then introduced, and we emphasize the important, but seemingly paradoxical, point that perturbations in the RW gauge can be considered to be gauge invariant expressions. First order perturbation theory is given in Sec. III, starting in Sec. IIIA, with the presentation of the "wave equations," that is, the RW equation [1] and the Zerilli equation [2], that are used to evolve perturbations in linearized theory. Sec. IIIB discusses the problem of extracting outgoing wave amplitudes from the solution of the wave equations, and partially establishes the pattern that will be used for second order work. The second order wave equation, a "second order Zerilli equation," is developed in Sec. IVA and is shown, as are all the second order equations, to be similar to first order equations except for the inclusion of "source" terms quadratic in first order perturbations. The problem of extracting second order outgoing wave amplitude, and the gauge issues involved, are taken up in Sec. IVB. Lastly, a summary is given in Sec. V, along with a mention of some related issues.

A word about our notational conventions is in order. We will use coordinates $\{t, r, \theta, \phi\} = \{x^0, x^1, x^2, x^3\}$ always to have the meaning of Schwarzschild coordinates, in the limit of small perturbations. Greek indices, $\alpha, \beta, \lambda, \tau, \mu, \nu \ldots$ will refer to spacetime coordinates, and Latin indices $i, j, k, \ldots$, will be spatial indices on a constant time surface. Throughout the paper we will use a consistent scheme for describing properties of our perturbation functions. A tilde ($\tilde{}$) denotes that a perturbation quantity is in a general gauge, while "RW" and "AF," as superscripts to the right of a symbol, mean that the quantity is defined in the Regge–Wheeler [1] or asymptotically flat gauge. Superscripts in parenthesis to the left of a quantity will indicate whether the quantity is a first or second order quantity. Superscripts in parentheses, to the right of a symbol will (sometimes) be used to denote multipole indices. A numerical subscript to the right of a quantity will be used, following the notation of Regge and Wheeler [1], as part of the name of various perturbation quantities. To distinguish even and odd parity Regge–Wheeler perturbations, we shall always explicitly add a right superscript "odd" to the odd parity perturbations; the omission of the "odd" indicates that the perturbation is even. Thus, for example, The quantity ${}^{(2)}h_1^{\mathrm{AF}\,(3,0)}$ indicates the axisymmetric octupolar ($\ell = 3, m = 0$) part of the second order perturbation of the Regge–Wheeler even parity quantity h_1, in an asymptotically flat gauge. For compatibility with notation in other papers, and for other purposes, we shall occasionally add further information in the form of indices and other attachments to symbols. This makes for notation that is sometimes very cumbersome and is often redundant, but we have found this a price well worth paying. In a mathematical description in which first and second order quantities, and quantities in different gauges, are used, all with similar symbols, clarity of meaning of the symbols is crucial. To reduce the notation we shall usually omit the multipole indices. Of the various indices, these seem to us to be the ones that are clearest from context.

On general notational choices we shall in follow the conventions of Misner, Thorne and Wheeler [22] and shall use the sign convention -+++ for the metric, and units in which $c = G = 1$. A dot over an index shall indicate partial

differentiation with respect to t. A subscript following a comma will indicate partial differentiation. For a function of a single argument a prime will indicate differentiation with respect to that argument.

II. PERTURBATION EXPANSIONS IN GENERAL

A. Basic issues

Our understanding of perturbation theory in general relativity is based on the idea of $g_{\alpha\beta}(x^\nu;\epsilon)$ a family of spacetime metrics parameterized by some physical quantity ϵ. For our family of spacetimes $g_{\alpha\beta}(x^\nu;\epsilon)$, we suppose that, at least in some restricted region of spacetime, the metric functions can be written as

$$g_{\alpha\beta}(x^\nu;\epsilon) = {}^{(0)}g_{\alpha\beta}(x^\nu) + \epsilon\ {}^{(1)}g_{\alpha\beta}(x^\nu) + \epsilon^2\ {}^{(2)}g_{\alpha\beta}(x^\nu)\cdots. \tag{1}$$

The metric ${}^{(0)}g_{\alpha\beta}(x^\nu)$ is the "background" metric, some known solution of Einstein's equations. The expanded form of the metric in (1) can be substituted in the Einstein equations, and the resulting set of equations can be expanded to orders in ϵ. The equations can then in principle be solved order by order, first for ${}^{(1)}g_{\alpha\beta}(x^\nu)$, (first-order perturbation theory) then for ${}^{(2)}g_{\alpha\beta}(x^\nu)$, and so forth.

There is an alternative way of viewing first order perturbation theory. One supposes that there is a *particular* spacetime $g^{\rm part}_{\alpha\beta}(x^\nu)$ that is in some sense close to a known "background" solution ${}^{(0)}g_{\alpha\beta}$ of Einstein's equations. One writes

$$g_{\alpha\beta}(x^\nu;\epsilon) \equiv {}^{(0)}g_{\alpha\beta}(x^\nu) + \epsilon\left[g^{\rm part}_{\alpha\beta}(x^\nu) - {}^{(0)}g_{\alpha\beta}(x^\nu)\right]\ . \tag{2}$$

and substitutes the right hand side of (2) into Einstein's equations, and treats ϵ as a formal parameter for keeping track of orders of perturbations. Finally, one sets ϵ to unity. This method turns out to give the correct equations for treating $g^{\rm part}_{\alpha\beta}(x^\nu)$ as a first order perturbation of ${}^{(0)}g_{\alpha\beta}$, but it is not based on a systematic approach to perturbation theory. For higher order computations, such a systematic approach is important.

A very relevant example of a family of spacetimes is the Misner spacetimes, representing two equal mass, initially stationary throats. The 3-geometry for this initial value solution is given by

$$ds^2 = a^2\widehat{\Phi}^4\left[d\mu^2 + d\eta^2 + \sin^2\eta\, d\varphi^2\right] \tag{3}$$

where

$$\widehat{\Phi} \equiv \sum_{n=-\infty}^{+\infty}\frac{1}{\sqrt{\cosh(\mu+2n\mu_0)-\cos(\eta)}}\ . \tag{4}$$

We now introduce new coordinates R,θ,ϕ, related to μ,η,ϕ in the same way that spherical polar coordinates are related to bispherical coordinates in flat space:

$$\cosh^2\mu = \frac{\left(R^2+a^2\right)^2}{\left(R^2+a^2\right)^2-\left(2aR\cos\theta\right)^2}\qquad \cos^2\eta = \frac{\left(R^2-a^2\right)^2}{\left(R^2-a^2\right)^2+\left(2aR\sin\theta\right)^2}\ . \tag{5}$$

The metric as presented appears in "isotropic" (conformally flat) form. In order to recover in the close limit the usual form of the Schwarzschild solution we transform to a new radial coordinate r through the usual transformation,

$$R = \frac{1}{4}\left(\sqrt{r}+\sqrt{r-2M}\right)^2\ , \tag{6}$$

where following Misner we have defined $M \equiv 4a\Sigma_1$ where

$$\Sigma_1 \equiv \sum_{n=1}^{\infty}\frac{1}{\sinh n\mu_0}\ . \tag{7}$$

With this new coordinate the 3-geometry takes the form

$$ds^2_{\rm Misner} = \mathcal{F}(r,\theta)^4\left(\frac{dr^2}{1-2M/r}+r^2d\Omega^2\right)\ , \tag{8}$$

with

$$\mathcal{F} \equiv 1 + 2\,(1 + \frac{M}{2R})^{-1} \sum_{\ell=2,4\ldots} \kappa_\ell(\mu_0)\,(M/R)^{\ell+1}\,P_\ell(\cos\theta)\ , \tag{9}$$

and with the κ_ℓ coefficients given by

$$\kappa_\ell(\mu_0) \equiv \frac{1}{(4\Sigma_1)^{\ell+1}} \sum_{n=1}^{\infty} \frac{(\coth n\mu_0)^\ell}{\sinh n\mu_0}\ . \tag{10}$$

In the limit $\mu_0 \to 0$, all κ_ℓ coefficients vanish, so the Misner initial geometry (more properly, that part of it covered by $r > 2M$) approaches the 3-geometry of a constant t slice of the Schwarzschild spacetime. But parameterization for perturbation theory of the Misner geometry is nontrivial. The "obvious" choice of parameter μ_0 cannot be used, because the deviations from the Schwarzschild geometry are not linear in μ_0 at small μ_0. The spacetime *can* be expanded in the parameter $\delta \equiv 1/\ln(\mu_0)$. The spacetime will then have nonzero perturbations at order $\delta^3, \delta^5, \delta^6, \delta^7, \delta^8, \cdots$. If, however, we project out only the quadrupole parts of the deviations from the Schwarzschild geometry, the leading terms in the expansion are of order $\delta^3, \delta^6, \delta^8, \delta^9, \cdots$. For consideration of the two lowest orders in this expansion it is useful to define $\epsilon \equiv \kappa_2(\mu_0) = \mathcal{O}(\delta^3)$, and to treat the problem as if we were doing perturbation to first and second order in ϵ.

1. Reparameterization

The expansion in (1) is complicated by several issues of choice. One of these is simply the question of the choice of parameter [23]. One could choose a new parameter by $\epsilon = \mathcal{E}[\epsilon']$, and treat ϵ' as the parameter, so that

$$g_{\alpha\beta}(x^\nu; \mathcal{E}[\epsilon']) = {}^{(0)}g_{\alpha\beta}(x^\nu) + \epsilon'\ {}^{(1)}g'_{\alpha\beta}(x^\nu) + \epsilon'^2\ {}^{(2)}g'_{\alpha\beta}(x^\nu) \cdots\ . \tag{11}$$

The condition for the spacetime to have a Taylor expansion in both ϵ and ϵ', is that ϵ can be expanded as

$$\epsilon = A\epsilon' + B\epsilon'^2 + \cdots. \tag{12}$$

For clarity of explanation, here we will restrict ourselves to transformations with $A = 1$. (This involves no real loss of generality, since it can always be accomplished with a trivial multiplicative rescaling.) With this restriction we have ${}^{(1)}g'_{\alpha\beta}(x^\nu) = {}^{(1)}g_{\alpha\beta}(x^\nu)$. The implication is that the first-order prediction for a given numerical value of ϵ (say $\epsilon = 0.1$) is identical to the first order prediction for the same value of ϵ' (say $\epsilon' = 0.1$). Since we are free to choose the function $\mathcal{E}$, let us suppose that we choose it such that $\mathcal{E}[0.1] = 10$, say from the reparameterization $\epsilon = \epsilon'/(1 + 9.9\epsilon')$. This means that $\epsilon = 0.1$ and $\epsilon' = 10$ correspond to the same physical situation, say the same initial separation between two coalescing holes. In the limit as ϵ gets very small, of course, the nature of the transformation in (12) guarantees that the first-order perturbation method answers will become insensitive to the choice of parameterization. But that fact may be misleading, since in practice one does perturbation calculations for specific physical situations, not as a limit. Let us suppose that $\epsilon = 0.1$ represents a physical situation and a parameterization for which first order theory gives good results. Then our example of a transformation to $\epsilon' = 10$, leads to inaccurate results despite the fact that the underlying physical problem was amenable to a perturbative solution. This illustrates the important point that the full nonlinear nature of a choice, like that of parameterization, can have an important effect on the results of first order calculations.

2. Gauge transformations

Special attention is paid to perturbative coordinate transformations, that is, to transformations that can be written as

$$x'^\mu = x^\mu + \epsilon\ {}^{(1)}\xi^\mu + \epsilon^2\ {}^{(2)}\xi^\mu + \cdots. \tag{13}$$

where ${}^{(1)}\xi^\mu$, ${}^{(2)}\xi^\mu, \cdots$ are functions of x^μ. The transformation in (13) induces a transformation of tensor fields. If (suppressing all indices) we let T represent any tensor then the first order transformation is given by

$$T' = T - \mathcal{L}_{\ {}^{(\vec{1})}\xi} T\ . \tag{14}$$

where $\mathcal{L}_{\,^{(\vec{1})}\xi}$ indicates the Lie derivative [24] taken with respect to the vector $^{(\vec{1})}\xi$. It is well known that such "gauge" transformations play an important role in first order perturbation calculations. Physical answers cannot depend on coordinate choices, so the coordinate freedom (gauge freedom) inherent in the gauge transformations must not affect physical results. In practice, this can be dealt with by constructing quantities which are invariant under gauge transformations, and then using Einstein's equations to compute these quantities. More typically, the problem is handled by "gauge fixing," i.e., by imposing specific restrictions on $^{(1)}g_{\alpha\beta}$ thereby fixing the coordinate system to first order.

For higher order perturbation computations the basic ideas are the same but there are crucial differences in detail. First, we remark that the functions $^{(2)}\xi^\mu$ in (13) are not vector fields, so the transformation induced on tensor fields cannot directly be represented by a geometric operation like that in (14). In our second order perturbation computations we deal with this by using a two-step process for gauge transformations.

In the first step the transformation $x^{\mu'} = x^\mu + \epsilon\ ^{(1)}\xi^\mu$ is made, and is carried out to (at least) second order. For a vector field V^μ, for example, the first step is the evaluation of the transformed components by

$$V^{\mu'}(x^\alpha) = \left(\delta^\mu_\alpha + \epsilon\ ^{(1)}\xi^\mu_{,\alpha}\right) V^\alpha(x^{\nu'} - \epsilon\ ^{(1)}\xi^\nu)\ . \tag{15}$$

The terms on the right can be evaluated exactly, but since we shall end by throwing away everything of order ϵ^3 or higher, we need only keep terms of order ϵ^2. If only these terms are kept the result is

$$\delta V^\alpha \equiv V^{\alpha'} - V^\alpha = -\epsilon\left(\mathcal{L}_{\,^{(\vec{1})}\xi}V\right)^\alpha + \tfrac{1}{2}\epsilon^2\ ^{(1)}\xi^\mu\ ^{(1)}\xi^\nu V^\alpha_{,\mu\nu} - \epsilon^2\ ^{(1)}\xi^\nu\ ^{(1)}\xi^\mu_\alpha V^\alpha_{,\nu}\ . \tag{16}$$

It is of no practical consequence that the ϵ^2 terms do not have a compact geometrical form. It is possible, however, to achieve a geometrical form by replacing our simple coordinate transformation $x^{\mu'} = x^\mu + \epsilon\ ^{(1)}\xi^\mu$ by

$$x^{\mu'} = x^\mu + \epsilon\ ^{(1)}\xi^\mu - \tfrac{1}{2}\epsilon^2\ ^{(1)}\xi^\sigma\ ^{(1)}\xi^\mu_{,\sigma}\ . \tag{17}$$

We find that, to second order in ϵ, the induced transformation for any tensor field V is

$$\delta V = -\epsilon\left(\mathcal{L}_{\,^{(\vec{1})}\xi}V\right) + \tfrac{1}{2}\epsilon^2\mathcal{L}_{\,^{(\vec{1})}\xi}\left(\mathcal{L}_{\,^{(\vec{1})}\xi}V\right)\ . \tag{18}$$

We shall not follow this path here; because of our two step procedure it really makes little difference. What is important in the first step is only that all tensor quantities are transformed correctly to second order in ϵ. If the first order transformations were not carried out correctly to order ϵ^2, then our tensor fields would change to that order. The spacetime geometry, in particular, would not be isometric to the original spacetime geometry. Though it is important that our first order transformation be correct to second order, the nature of the second order coordinate change, whether it is that of (17), or the first two terms on the right in (13), is irrelevant. The second step will obviate any specific choice.

We call step 1 the "first order gauge transformation" (though it must be carried out correctly to at least second order). The purpose of this step will typically be to impose some "first order gauge condition," that is, some conditions on the first order metric perturbations $^{(1)}g_{\alpha\beta}$ that can be brought about by the proper choice of $^{(1)}\xi^\mu$.

The second step in our procedure is to use a coordinate transformation of the form

$$x^{\mu'} = x^\mu + \epsilon^2\ ^{(2)}\xi^\mu\ , \tag{19}$$

and to transform only to lowest order, i.e., to transform tensors by

$$\delta V = -\epsilon^2\left(\mathcal{L}_{\,^{(\vec{2})}\xi}V\right)\ . \tag{20}$$

This transformation does not change the first order parts of tensor fields, and hence leaves intact the first order gauge conditions imposed by the transformation in our first step. With the second step we can use the choice of the fields $^{(2)}\xi$ to impose conditions on $^{(2)}g_{\alpha\beta}$. Note that the details of the functions $^{(2)}\xi^\mu$ needed to impose these second order gauge restrictions will depend on the second order part of the metric *after* the first step is performed. The metric, to second order is affected by the first step. This means that the details of the second step depend on the details of the first step, and this is why we do not need to be specific about the second order part of the coordinate transformation used in the first step.

Our two-step procedure in which we first make the transformation $x^{\mu'} = x^\mu + \epsilon\ ^{(1)}\xi^\mu$ and then $x^{\mu'} = x^\mu + \epsilon^2\ ^{(2)}\xi^\mu$ leads to the following explicit transformations of the metric perturbations.

$$
{}^{(1)}g'_{\mu\nu} = {}^{(1)}g_{\mu\nu} - {}^{(0)}g_{\mu\nu,\rho}\ {}^{(1)}\xi^{\rho} - {}^{(0)}g_{\mu\rho}\ {}^{(1)}\xi^{\rho}{}_{,\nu} - {}^{(0)}g_{\rho\nu}\ {}^{(1)}\xi^{\rho}{}_{,\mu} \tag{21}
$$

and,

$$
\begin{aligned}
{}^{(2)}g'_{\mu\nu} = {} & {}^{(2)}g_{\mu\nu} \\
& - {}^{(1)}g'_{\mu\nu,\rho}\ {}^{(1)}\xi^{\rho} - {}^{(1)}g'_{\mu\rho}\ {}^{(1)}\xi^{\rho}{}_{,\nu} - {}^{(1)}g'_{\rho\nu}\ {}^{(1)}\xi^{\rho}{}_{,\mu} \\
& - {}^{(0)}g_{\mu\nu,\rho}\ {}^{(2)}\xi^{\rho} - {}^{(0)}g_{\mu\rho}\ {}^{(2)}\xi^{\rho}{}_{,\nu} - {}^{(0)}g_{\rho\nu}\ {}^{(2)}\xi^{\rho}{}_{,\mu} \\
& - \frac{1}{2}\ {}^{(0)}g_{\mu\nu,\sigma,\lambda}\ {}^{(1)}\xi^{\sigma}\ {}^{(1)}\xi^{\lambda} - {}^{(0)}g_{\mu\lambda,\sigma}\ {}^{(1)}\xi^{\sigma}\ {}^{(1)}\xi^{\lambda}{}_{,\nu} \\
& - {}^{(0)}g_{\lambda\nu,\sigma}\ {}^{(1)}\xi^{\sigma}\ {}^{(1)}\xi^{\lambda}{}_{,\mu} - {}^{(0)}g_{\sigma\lambda}\ {}^{(1)}\xi^{\sigma}{}_{,\mu}\ {}^{(1)}\xi^{\lambda}{}_{,\nu}\ .
\end{aligned} \tag{22}
$$

Throughout this paper we shall use the two step procedure for second order gauge transformations. An alternative treatment has recently been presented by Bruni *et al.* [25]. That treatment gives a much more geometrical view of second order gauge transformations.

4. Standard hierarchy

For definiteness we shall now consider specifically the vacuum Einstein equations, which is in fact the case of interest for the close-limit method. Einstein's vacuum equations can be written as $\mathcal{G}_{\lambda\tau}\left(g_{\alpha\beta}(x^\nu)\right) = 0$, where $\mathcal{G}_{\lambda\tau}$ represents the actions of combining zeroth, first and second derivatives of $g_{\alpha\beta}$ to form the component $G_{\lambda\tau}$ of the Einstein tensor. If the expansion in (1) is used, the Einstein equations for the family of solutions

$$
\mathcal{G}_{\lambda\tau}\left(g_{\alpha\beta}(x^\nu;\epsilon)\right) = \mathcal{G}_{\lambda\tau}\left({}^{(0)}g_{\alpha\beta}(x^\nu) + \epsilon\ {}^{(1)}g_{\alpha\beta}(x^\nu) + \epsilon^2\ {}^{(2)}g_{\alpha\beta}(x^\nu) \cdots \right) = 0\ , \tag{23}
$$

can be expanded in powers of ϵ. The terms to order 0 in ϵ are just those of $\mathcal{G}\left({}^{(0)}g_{\alpha\beta}(x^\nu)\right) = 0$, and are satisfied immediately since, by assumption, ${}^{(0)}g_{\alpha\beta}$ is a solution to Einstein's equations.

The terms in (23) of first order in ϵ can be written in the form

$$
\epsilon L_{\lambda\tau}(\ {}^{(1)}g_{\alpha\beta}) = 0\ , \tag{24}
$$

where $L_{\lambda\tau}$, formally defined by

$$
L_{\lambda\tau} \equiv \frac{\partial}{\partial\epsilon}\mathcal{G}_{\lambda\tau}\left(g_{\alpha\beta}(x^\nu;\epsilon)\right)\Big|_{\epsilon=0} \tag{25}
$$

is a linear operator on ${}^{(1)}g_{\alpha\beta}$, consisting of combinations of differentiation and multiplications by specific coordinate functions. The details of L depend on the background solution ${}^{(0)}g_{\alpha\beta}$. The equations for ${}^{(1)}g_{\alpha\beta}$ contained in (24), constitute first order perturbation theory.

The part of (23) that is proportional to ϵ^2 has terms of two different types. There will be terms linear in ${}^{(2)}g_{\alpha\beta}$ and terms quadratic in ${}^{(1)}g_{\alpha\beta}$. The terms of the first type occur in precisely the same way as do the ${}^{(1)}g_{\alpha\beta}$ terms in the first order expression. We can therefore write the second order part of (23) as

$$
\epsilon^2 L_{\lambda\tau}(\ {}^{(2)}g_{\alpha\beta}) = \epsilon^2\ {}^{(2)}\mathcal{T}_{\lambda\tau}\left({}^{(1)}g_{\alpha\beta}\right)\ , \tag{26}
$$

where ${}^{(2)}\mathcal{T}_{\lambda\tau}$ is quadratic in ${}^{(1)}g_{\alpha\beta}$. One views (26) as a set of linear equations for ${}^{(2)}g_{\alpha\beta}$, with the right hand side a "source" which is known from the solution of the first order problem. In a similar manner one goes on to find that terms in (23) higher order in ϵ have the form

$$
\epsilon^n L^{(n)}_{\lambda\tau}(g_{\alpha\beta}) = \epsilon^n\ {}^{(n)}\mathcal{T}_{\lambda\tau}\left({}^{(n-1)}g_{\alpha\beta},\ {}^{(n-2)}g_{\alpha\beta},\ {}^{(n-3)}g_{\alpha\beta},\cdots\right)\ . \tag{27}
$$

In the $n^{\rm th}$ order source term ${}^{(n)}\mathcal{T}_{\lambda\tau}$ the combinations of the lower order metric perturbations must occur according to obvious rules. For example, the source ${}^{(5)}\mathcal{T}_{\lambda\tau}$, for the fifth order perturbations, will in general have contributions including terms of fifth power in ${}^{(1)}g_{\alpha\beta}$, terms of second power in ${}^{(1)}g_{\alpha\beta}$ multiplied by terms linear in ${}^{(3)}g_{\alpha\beta}$, and so forth. (Here "terms in ${}^{(k)}g_{\alpha\beta}$" includes derivatives of these terms.) In principle, one can solve order by order since

the source terms for each order are given by the solution known from the lower orders. Furthermore, at each order the linear operator being solved is precisely the same $L_{\lambda\tau}$; all that is changing is the source terms and the initial conditions.

There are alternatives to this "standard hierarchy" of equations for perturbations of increasing order. One could, for example, adopt the following alternative iterative scheme. Start with the equation

$$\mathcal{G}\left({}^{(0)}g_{\alpha\beta}(x^\nu) + \epsilon\ {}^{(1)}g_{\alpha\beta}(x^\nu) + \epsilon^2\ {}^{(2)}g_{\alpha\beta}(x^\nu)\cdots \right) = 0. \tag{28}$$

As a first step, keep only the terms that are first order in ϵ. This will give a set of linear equations for ${}^{(1)}g_{\alpha\beta}$ identical to (24). Next, in (28), substitute the known solution for ${}^{(1)}g_{\alpha\beta}$, omit ${}^{(k)}g_{\alpha\beta}$ terms for $k > 2$ and solve the resulting equations to lowest order in ${}^{(2)}g_{\alpha\beta}$. The resulting linear equations for ${}^{(2)}g_{\alpha\beta}$ will *not* be the same as those in (26). In the present method we are in effect feeding back *all* the information about the metric resulting from the first order solution, and we are therefore keeping different terms that are higher oder in ϵ. In this alternative method, the differential equation for ${}^{(2)}g_{\alpha\beta}$ represents propagation of the second order perturbations on a background spacetime correct to first order. In the standard hierarchy the second order perturbations, and all perturbations, propagate on the zero order background.

This alternative method would seem intuitively to offer advantages, and indeed is being investigated, but it entails a very serious practical difficulty. In the standard hierarchy, the differential operator L embodies the simplicity and symmetries of the background solution. In practice this means, for example, that multipole decomposition for angular variables can be used. In the alternative method the differential operator for higher order perturbations no longer has those simplifications. The higher order equations, as in the standard hierarchy, are linear, but they would need to be solved numerically as 2+1 or 3+1 linear hyperbolic systems.

B. Schwarzschild perturbations

We specialize now to a spherically symmetric background and introduce coordinates $\{x^0, x^1, x^2, x^3\} = \{t, r, \theta, \phi\}$ chosen such that in the $\epsilon \to 0$ limit θ, ϕ become the usual spherical coordinates. The "background operator" L acting on the unknown perturbations embodies the spherical symmetry of the background, and so allows us to eliminate angular variables θ, ϕ. To take advantage of the symmetry it is necessary to expand the metric perturbations in tensor spherical harmonics. This does not require that we make further assumptions about the background geometry, but below we shall specialize to the case of a Schwarzschild geometry. To avoid the repetition of very similar lengthy expressions, we give here the description of the multipole decomposition specific to the Schwarzschild background, where we follow the notations and conventions [1] of Regge and Wheeler (RW).

Our background metric ${}^{(0)}g_{\mu\nu}$ is the standard exterior Schwarzschild solution in Schwarzschild's coordinates

$$a = \begin{matrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{matrix} . \tag{29}$$

The 10 metric perturbations can be divided into two sets, called perturbations of odd and even parity, which are not mixed by tensorial operators which respect spherical symmetry. The multipole decomposition, is given below for even parity perturbations

$$\begin{aligned}
{}^{(n)}g_{00} &= \left(1 - \frac{2M}{r}\right)\sum_{\ell,m}\ {}^{(n)}\widetilde{H_0}^{(\ell,m)} Y_\ell^m \\
{}^{(n)}g_{01} &= \sum_{\ell,m}\ {}^{(n)}\widetilde{H_1}^{(\ell,m)} Y_\ell^m \\
{}^{(n)}g_{02} &= \sum_{\ell,m}\ {}^{(n)}\widetilde{h_0}^{(\ell,m)} \frac{\partial Y_\ell^m}{\partial\theta} \\
{}^{(n)}g_{03} &= \sum_{\ell,m}\ {}^{(n)}\widetilde{h_0}^{(\ell,m)} \frac{\partial Y_\ell^m}{\partial\phi} \\
{}^{(n)}g_{11} &= \sum_{\ell,m}\left(1 - \frac{2M}{r}\right)^{-1}\ {}^{(n)}\widetilde{H_2}^{(\ell,m)} Y_\ell^m
\end{aligned}$$

$$
\begin{aligned}
{}^{(n)}g_{12} &= \sum_{\ell,m} {}^{(n)}\widetilde{h_1}^{(\ell,m)} \frac{\partial Y_\ell^m}{\partial\theta} \\
{}^{(n)}g_{13} &= \sum_{\ell,m} {}^{(n)}\widetilde{h_1}^{(\ell,m)} \frac{\partial Y_\ell^m}{\partial\phi} \\
{}^{(n)}g_{22} &= r^2 \sum_{\ell,m} \left({}^{(n)}\widetilde{K}^{(\ell,m)} + {}^{(n)}G^{(\ell,m)} \frac{\partial^2}{\partial\theta^2} \right) Y_\ell^m \\
{}^{(n)}g_{23} &= r^2 \sum_{\ell,m} {}^{(n)}\widetilde{G}^{(\ell,m)} \left(\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} \right) Y_\ell^m \\
{}^{(n)}g_{33} &= r^2 \sum_{\ell,m} \left[{}^{(n)}\widetilde{K}^{(\ell,m)} \sin^2(\theta) + {}^{(n)}\widetilde{G}^{(\ell,m)} \left(\frac{\partial^2}{\partial\phi^2} + \sin\theta\cos\theta\frac{\partial}{\partial\theta} \right) \right] Y_\ell^m \ ,
\end{aligned}
\tag{30}
$$

and odd parity perturbations

$$
\begin{aligned}
{}^{(n)}g_{02\mathrm{odd}} &= -\sum_{\ell,m} \frac{{}^{(n)}\widetilde{h_0}^{(\ell,m)\mathrm{odd}}}{\sin\theta} \frac{\partial Y_\ell^m}{\partial\phi} \\
{}^{(n)}g_{03\mathrm{odd}} &= \sum_{\ell,m} {}^{(n)}\widetilde{h_0}^{(\ell,m)\mathrm{odd}} \sin\theta \frac{\partial Y_\ell^m}{\partial\theta} \\
{}^{(n)}g_{12\mathrm{odd}} &= -\sum_{\ell,m} \frac{{}^{(n)}\widetilde{h_1}^{(\ell,m)\mathrm{odd}}}{\sin\theta} \frac{\partial Y_\ell^m}{\partial\phi} \\
{}^{(n)}g_{13\mathrm{odd}} &= \sum_{\ell,m} {}^{(n)}\widetilde{h_1}^{(\ell,m)\mathrm{odd}} \sin\theta \frac{\partial Y_\ell^m}{\partial\theta} \\
{}^{(n)}g_{22\mathrm{odd}} &= \sum_{\ell,m} {}^{(n)}\widetilde{h_2}^{(\ell,m)\mathrm{odd}} \left(\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} \right) Y_\ell^m \\
{}^{(n)}g_{33\mathrm{odd}} &= -\sum_{\ell,m} {}^{(n)}\widetilde{h_2}^{(\ell,m)\mathrm{odd}} \left(\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi} \right) Y_\ell^m \\
{}^{(n)}g_{23\mathrm{odd}} &= \frac{1}{2}\sum_{\ell,m} {}^{(n)}\widetilde{h_2}^{(\ell,m)\mathrm{odd}} \left(\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial^2}{\partial\theta^2} \right) Y_\ell^m \ ,
\end{aligned}
\tag{31}
$$

where $\widetilde{H_0}^{\ell,m)}$, $\widetilde{H_1}^{(\ell,m)}$, $\widetilde{H_2}^{(\ell,m)}$, $\widetilde{h_0}^{(\ell,m)}$, $\widetilde{h_1}^{(\ell,m)}$, $\widetilde{K}^{(\ell,m)}$ and $\widetilde{G}^{(\ell,m)}$, are functions of r and t for the even parity parts of the perturbations, and ${}^{(n)}\widetilde{h}_0^{(\ell,m,\mathrm{odd})}$, ${}^{(n)}\widetilde{h}_1^{(\ell,m,\mathrm{odd})}$, ${}^{(n)}\widetilde{h}_2^{(\ell,m),\mathrm{odd}}$ are the odd parity functions. The complete metric perturbations are the sum of the even and the odd parity parts. Here we introduce the notation that a tilde $\tilde{}$ over a perturbation function indicates that it is in an arbitrary gauge. As an example of metric perturbations we have, from (8) and (9), the initial perturbations of the Misner spacetime. For $\ell = 2$, these are

$$
{}^{(1)}\widetilde{H}_2 = {}^{(1)}\widetilde{K} = 8\kappa_2 \left(1 - \frac{2M}{R}\right)^{-1} \left(\frac{2M}{R}\right)^3 \sqrt{\frac{4\pi}{5}} \ , \tag{32}
$$

and ${}^{(1)}\widetilde{G} = {}^{(1)}\widetilde{h}_1 = 0$. We can choose to set the initial lapse and shift such that ${}^{(1)}\widetilde{H}_0 = {}^{(1)}\widetilde{H}_1 = {}^{(1)}\widetilde{h}_0 = 0$.

We now fix coordinates to first order by demanding that the perturbations satisfy the RW gauge conditions: For even parity this is the condition that the first order even parity functions ${}^{(1)}h_0^{(\ell,m)}$, ${}^{(1)}h_1^{(\ell,m)}$, ${}^{(1)}G^{(\ell,m)}$ vanish. The odd parity RW gauge choice is that the function ${}^{(1)}h_2^{(\ell,m)\mathrm{odd}}$ vanishes. This specialization is accomplished with the first order gauge transformation (21), with the following notation:

$$ {}^{(1)}\xi^0 = {}^{(1)}A_0 Y_\ell^m \tag{33} $$
$$ {}^{(1)}\xi^1 = {}^{(1)}A_1 Y_\ell^m \tag{34} $$
$$ {}^{(1)}\xi^2 = {}^{(1)}A_2 Y_\ell^m{}_{,\theta} + {}^{(1)}B^{\mathrm{odd}} Y_\ell^m{}_{,\phi} / \sin\theta \tag{35} $$
$$ {}^{(1)}\xi^3 = {}^{(1)}A_2 Y_\ell^m{}_{,\phi} / \sin^2\theta - {}^{(1)}B^{\mathrm{odd}} Y_\ell^m{}_{,\theta} / \sin\theta \tag{36} $$

These gauge functions give transformations of the RW metric perturbation functions as follows

$$
\begin{aligned}
{}^{(1)}h_0^{\rm RW,odd} &= {}^{(1)}\widetilde{h}_0^{\rm odd} + r^2 \frac{\partial\, {}^{(1)}B^{\rm odd}}{\partial t} \\
{}^{(1)}h_1^{\rm RW,odd} &= {}^{(1)}\widetilde{h}_1^{\rm odd} + r^2 \frac{\partial\, {}^{(1)}B^{\rm odd}}{\partial r} \\
{}^{(1)}h_2^{\rm RW,odd} &= {}^{(1)}h_2^{\rm odd} - 2r^2\, {}^{(1)}B^{\rm odd} \\
{}^{(1)}H_0^{\rm RW} &= {}^{(1)}\widetilde{H}_0 + \frac{2M}{r(r-2M)}\, {}^{(1)}A_1 + 2\frac{\partial\, {}^{(1)}A_0}{\partial t} \\
{}^{(1)}H_1^{\rm RW} &= {}^{(1)}\widetilde{H}_1 - \frac{r}{r-2M}\frac{\partial\, {}^{(1)}A_1}{\partial t} + \frac{r-2M}{r}\frac{\partial\, {}^{(1)}A_0}{\partial r} \\
{}^{(1)}h_0^{\rm RW} &= {}^{(1)}\widetilde{h}_0 - r^2\frac{\partial\, {}^{(1)}A_2}{\partial t} + \frac{r-2M}{r}\, {}^{(1)}A_0 \\
{}^{(1)}H_2^{\rm RW} &= {}^{(1)}\widetilde{H}_2 + \frac{2M}{r(r-2M)}\, {}^{(1)}A_1 - 2\frac{\partial\, {}^{(1)}A_1}{\partial r} \\
{}^{(1)}h_1^{\rm RW} &= {}^{(1)}\widetilde{h}_1 - \frac{r}{r-2M}\, {}^{(1)}A_1 - r^2\frac{\partial\, {}^{(1)}A_2}{\partial r} \\
{}^{(1)}G^{\rm RW} &= {}^{(1)}\widetilde{G} - 2\, {}^{(1)}A_2 \\
{}^{(1)}K^{\rm RW} &= {}^{(1)}\widetilde{K} - \frac{2}{r}\, {}^{(1)}A_1\ .
\end{aligned} \tag{37}
$$

with the gauge functions

$$ {}^{(1)}A_0 = \left(\frac{1}{2}r^2\frac{\partial\, {}^{(1)}\widetilde{G}}{\partial t} - {}^{(1)}\widetilde{h_0}\right) \tag{38} $$

$$ {}^{(1)}A_1 = (1-2M/r)\left(-\frac{1}{2}r^2\frac{\partial\, {}^{(1)}\widetilde{G}}{\partial r} + {}^{(1)}\widetilde{h_1}\right) \tag{39} $$

$$ {}^{(1)}A_2 = \tfrac{1}{2}\, {}^{(1)}\widetilde{G} \tag{40} $$

$$ {}^{(1)}B^{\rm odd} = \frac{1}{2r^2}\, {}^{(1)}\widetilde{h_2}^{\rm odd}\ . \tag{41} $$

With this choice of first order gauge transformation, the result for the first order perturbations is:

$$ {}^{(1)}h_0^{\rm RW,odd} = \tilde{h}_0^{\rm odd} + \tfrac{1}{2}\, {}^{(1)}\widetilde{h_2}^{\rm odd}{}_{,t} \tag{42} $$

$$ {}^{(1)}h_1^{\rm RW,odd} = {}^{(1)}\tilde{h}_1^{\rm odd} + \tfrac{1}{2}r^2\left(r^{-2}\, {}^{(1)}\widetilde{h_2}^{\rm odd}\right)_{,r} \tag{43} $$

and those for the even-parity functions are

$$ {}^{(1)}K^{RW} = {}^{(1)}\widetilde{K} + (r-2M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\, {}^{(1)}\widetilde{h}_1\right) \tag{44} $$

$$ {}^{(1)}H_2^{RW} = {}^{(1)}\widetilde{H}_2 + (2r-3M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\, {}^{(1)}\widetilde{h}_1\right) + r(r-2M)\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\, {}^{(1)}\widetilde{h}_1\right)_{,r} \tag{45} $$

$$ {}^{(1)}H_1^{RW} = {}^{(1)}\widetilde{H}_1 + r^2\, {}^{(1)}\widetilde{G}_{,tr} - {}^{(1)}\widetilde{h}_{1,t} - \frac{2M}{r(r-2M)}\, {}^{(1)}\widetilde{h}_0 + {}^{(1)}\widetilde{h}_{0,r} + \frac{r(r-3M)}{r-2M}\, {}^{(1)}\widetilde{G}_{,t} \tag{46} $$

$$ {}^{(1)}H_0^{RW} = {}^{(1)}\widetilde{H}_0 - M\left({}^{(1)}\widetilde{G}_{,r} - \frac{2}{r^2}\, {}^{(1)}\widetilde{h}_1\right) + \frac{2r}{r-2M}\, {}^{(1)}\widetilde{h}_{0,t} + \frac{r^3}{(r-2M)}\, {}^{(1)}\widetilde{G}_{,tt}\ . \tag{47} $$

Here we have dropped the ℓ, m indices in order to simplify the intricate notation.

The above equations show that we can choose to view the left-hand quantities not as metric perturbations expressed in a particular coordinate gauge, but rather (due to the right hand side) as combinations of metric perturbations expressed in an arbitrary coordinate gauge. In this sense we can, and we will, view the RW-gauge quantities, such as

H_1^{RW} as a compact expression for a general-gauge expression. It should be emphasized that it is trivial, combining the above results to construct expressions that are *explicitly gauge invariant* in terms of the metric perturbations in an *arbitrary* gauge.

It is instructive to point out the somewhat special aspects of the RW gauge choice and to compare it to other approaches. Chandrasekhar [26] chooses to work in a diagonal metric, for axisymmetric perturbations, by making the even parity restrictions ${}^{(1)}H_1^{\ell m} = {}^{(1)}h_0^{\ell m} = {}^{(1)}h_1^{\ell m} = 0$. This choice turns out not to uniquely fix the gauge [27]. One can, for example, perform a nontrivial transformation (41) with ${}^{(1)}A_0 = 0$, with ${}^{(1)}A_2$ any function of r alone, and with ${}^{(1)}A_1$ given by

$$ {}^{(1)}A_1 = -r(r-2M)\frac{\partial\ {}^{(1)}A_2}{\partial r}\ . \tag{48} $$

Such a transformation leaves unchanged the values of ${}^{(1)}H_1^{\ell m}$, ${}^{(1)}h_0^{\ell m}$ and ${}^{(1)}h_1^{\ell m}$, but changes other perturbations. This means that the metric perturbations in the Chandrasekhar gauge are not unique, and thus it is impossible for them to be expressed uniquely in terms of arbitrary gauge metric perturbations, as the RW perturbations are in (44) – (47).

Another approach to perturbation theory is that of Moncrief [28] who, for analysis of even parity perturbations, uses gauge invariant combinations only of the 3-geometry quantities ${}^{(1)}\widetilde{H}_2^{\ell m}$, ${}^{(1)}\widetilde{K}^{\ell m}$, ${}^{(1)}\widetilde{G}^{\ell m}$ and ${}^{(1)}\widetilde{h}_1^{\ell m}$. This approach is quite useful for connecting the computation of radiation to initial value data, since the quantities used are, by construction, independent of the (necessarily arbitrary) lapse and shift. The Moncrief approach will be discussed again below. Here we confine our attention to the question of whether one could invoke a "Moncrief gauge," a gauge in which perturbations of the lapse and shift are set to zero, by choosing ${}^{(1)}H_1^{\ell m} = {}^{(1)}H_0^{\ell m} = {}^{(1)}h_0^{\ell m} = 0$. In such a gauge computations would only involve perturbations of the 3 geometry. It is easy to show, however, that such a choice suffers from the same problem as the Chandrasekhar gauge. The gauge conditions do not completely fix the gauge.

The transformation to the RW gauge (or the derivation of the RW arbitrary-gauge expressions) has been discussed as a first order problem. We now view this as the first step in the two step process we discussed above. Our second step is a second order gauge transformation [see (13)] of the form

$$ x^{\mu'} = x^\mu + \epsilon^2\ {}^{(2)}\xi^\mu . \tag{49} $$

We choose ${}^{(2)}\xi^\mu$ to impose the RW gauge conditions (e.g., ${}^{(2)}G = 0$) to second order. The form of ${}^{(2)}\xi^\mu$ needed is exactly that of ξ^μ in (33) through (41), with the index "1" replaced by "2". Thus for example, second order gauge transformations can be written as:

$$ {}^{(2)}\xi^0 = {}^{(2)}A_0 Y_\ell^m \tag{50} $$
$$ {}^{(2)}\xi^1 = {}^{(2)}A_1 Y_\ell^m \tag{51} $$
$$ {}^{(2)}\xi^2 = {}^{(2)}A_2 Y_\ell^m{}_{,\theta} + {}^{(2)}B^{\rm odd} Y_\ell^m{}_{,\phi} / \sin\theta \tag{52} $$
$$ {}^{(2)}\xi^3 = {}^{(2)}A_2 Y_\ell^m{}_{,\phi} / \sin^2\theta - {}^{(2)}B^{\rm odd} Y_\ell^m{}_{,\theta} / \sin\theta\ . \tag{53} $$

To set ${}^{(2)}H_1^{\ell m} = {}^{(2)}h_0^{\ell m} = {}^{(2)}h_1^{\ell m} = 0$ we use, for example, ${}^{(2)}A_2 = \frac{1}{2}\ {}^{(2)}\widetilde{G}$, and find relations between the second order RW perturbations and the second order perturbations in a general gauge, for example

$$ {}^{(2)}K^{RW} = {}^{(2)}\widetilde{K} + (r-2M)\left({}^{(2)}\widetilde{G}_{,r} - 2r^{-2}\ {}^{(2)}\widetilde{h}_1 \right) . \tag{54} $$

It should be mentioned that there appear at second order certain ambiguities in going to the RW gauge if the first order perturbations have $\ell = 0$ components. The gauge fixing procedure leaves a residual gauge symmetry that has to, and can be, dealt with. This issue has been discussed in detail in [20], and we will not address it here.

III. FIRST ORDER PERTURBATIONS

A. Wave equations

The first order part of Einstein's equations, the content of (24), constitutes linearized perturbation theory. Though this theory is long established, we review it here since it forms the foundation of our second order computations.

Even and odd parity perturbations completely decouple in first order theory and we can analyze them separately. It is useful to consider first the relatively simple odd parity problem, first solved in 1957 by Regge and Wheeler [1]. In the specialized RW gauge the nontrivial odd parity equations are

$$\frac{\partial^2\ {}^{(1)}h_0^{\rm RW}}{\partial r^2}-\frac{\partial^2\ {}^{(1)}h_1^{\rm RW}}{\partial r\partial t}-\frac{2}{r}\frac{\partial\ {}^{(1)}h_1^{\rm RW}}{\partial t}+\left[\frac{4M}{r}-\ell(\ell+1)\right]\frac{{}^{(1)}h_0^{\rm RW}}{r(r-2M)}=0 \tag{55}$$

$$\frac{\partial^2\ {}^{(1)}h_1^{\rm RW}}{\partial r^2}-\frac{\partial^2\ {}^{(1)}h_0^{\rm RW}}{\partial r\partial t}+(\ell-1)(\ell+2)(r-2M)\frac{{}^{(1)}h_1^{\rm RW}}{r^3}=0 \tag{56}$$

$$\left(1-\frac{2M}{r}\right)\frac{\partial\ {}^{(1)}h_1^{\rm RW}}{\partial r}-\left(1-\frac{2M}{r}\right)\frac{\partial\ {}^{(1)}h_0^{\rm RW}}{\partial t}+\frac{2M}{r^2}\ {}^{(1)}h_0^{\rm RW}=0\ . \tag{57}$$

Regge and Wheeler [1] defined, for each multipole, the wavefunction

$$Q^{\rm odd}\equiv r^{-1}(1-2M/r)h_1^{\rm RW,odd}\ . \tag{58}$$

The field equations above show that this perturbation quantity decouples from ${}^{(1)}h_0^{\rm RW,odd}$, and satisfies the wave equation

$$\frac{\partial^2 Q^{\rm odd}}{\partial r^{*2}}-\frac{\partial^2 Q^{\rm odd}}{\partial t^2}+Q^{\rm odd}V_\ell^{\rm RW}(r)=0\ . \tag{59}$$

Here $V_\ell^{\rm RW}$ is an ℓ-dependent function of r that acts as a potential in the wave equation, and $r^*\equiv r-2M\ln{(r/2M-1)}$ is the "tortoise" radial coordinate introduced by Regge and Wheeler. Once (59) is solved and $Q^{\rm odd}$, and hence $h_1^{\rm RW,odd}$ are known, one can find $h_0^{\rm RW,odd}$ by solving (57). In solving that equation, the integration constant is supplied by the specification of $h_0^{\rm RW,odd}$ on the initial $t=0$ hypersurface (which is related to the initial extrinsic curvature; see below). Once $h_1^{\rm RW,odd}$ and $h_0^{\rm RW,odd}$ are known, all gauge invariant odd-parity information is known.

The odd-parity wave equation (59) requires of course the specification of Cauchy data, and these must be supplied from the first order perturbations of the 3-metric γ_{ij} and the extrinsic geometry K_{ij} of an initial value solution. Here and throughout we will take our initial surface to be a surface at $t=0$. From initial information about the perturbed 3-metric ${}^{(1)}\gamma_{ij}$ in some gauge, we can immediately infer the value of ${}^{(1)}\widetilde{h_1}^{\rm odd}$ at $t=0$ and ${}^{(1)}\widetilde{h_2}^{\rm odd}$ at $t=0$. With these, and with (43), we can compute ${}^{(1)}{h_1}^{\rm RW,odd}$ at $t=0$ and hence can compute $Q^{\rm odd}$. The general relationship between the extrinsic curvature and the time derivative of the metric is

$$K_{ij}=\frac{1}{2N}\left[g_{0i|j}+g_{0j|i}-\frac{\partial}{\partial t}g_{ij}\right]\ . \tag{60}$$

Here N is the lapse function, given by $\sqrt{-1/g^{tt}}$. The initial value of the odd-parity shift, ${}^{(1)}\widetilde{g}^{\rm odd}_{t\theta}$, ${}^{(1)}\widetilde{g}^{\rm odd}_{t\phi}$ or equivalently ${}^{(1)}\tilde{h}_0^{\rm odd}$ can be freely specified, and it is convenient to set ${}^{(1)}\tilde{h}_0^{\rm odd}$ initially to zero. With this choice, we have

$$K_{ij}=\frac{1}{2}\sqrt{\frac{r}{r-2M}}\ \frac{\partial}{\partial t}g_{ij}. \tag{61}$$

The time derivative of ${}^{(1)}\tilde{h}_2^{\rm odd}$ follows from the odd parity part of the $\theta\theta,\phi\phi$, or $\theta\phi$ components of this equation. The time derivative of ${}^{(1)}\tilde{h}_1^{\rm odd}$ follows from the odd parity part of the $r\theta$ or $r\phi$ components. From these time derivatives, and from (43), we find the time derivative of ${}^{(1)}{h_1}^{\rm RWodd}$ and hence of $Q^{\rm odd}$.

Moncrief has taken a distinctly different approach. He works in an arbitrary gauge with only the perturbations of the 3-geometry. (Thus for odd parity he works with ${}^{(1)}\widetilde{h_1}^{\rm odd}$ and ${}^{(1)}\widetilde{h_2}^{\rm odd}$, but not ${}^{(1)}\widetilde{h_0}^{\rm odd}$.) From those quantities he constructs combinations which are invariant with respect to diffeomorphisms on the hypersurfaces. Since the quantities, by construction, are automatically independent of shift and lapse choice, they are totally gauge invariant. Perturbation quantities which decouple from others must have this property. Moncrief's method leads to the same quantity $Q^{\rm odd}$ as that derived by Regge and Wheeler [if $Q^{\rm odd}$ is interpreted in an arbitrary gauge with (43)].

In the case of even parity perturbations, the situation becomes much more complicated. Zerilli has derived a wave equation (the "Zerilli equation") for a single decoupled quantity, by working with Fourier transforms, that is, by assuming a time dependence $e^{-i\omega t}$. We shall not be able to use Fourier transforms when dealing with the nonlinear terms in second order perturbations, so we start by re-deriving the Zerilli wave equation in the time domain. A

point to notice is that Zerilli introduces a function R that is equivalent in the time domain to $i\int H_1\,dt$. To avoid introducing an integral we will be working with what amounts to the time derivative of Zerilli's equations. Our time domain equations can be compared to the frequency domain equations of Zerilli by replacing $\partial/\partial t$ in our expressions by $-i\omega$, and by replacing our ${}^{(1)}\widehat{R}$ with Zerilli's $\widehat{R}_{LM}$, and our ${}^{(1)}\chi$ by Zerilli's $-i\omega\widehat{K}_{LM}$.

From one of the vacuum Einstein equations we find that ${}^{(1)}H_2^{\rm RW} = {}^{(1)}H_0^{\rm RW}$. The remaining equations can be broken into a first set that contains only first order derivatives in r and that can be written in the form

$$
\begin{aligned}
\frac{\partial\,{}^{(1)}H_1^{\rm RW}}{\partial r} &= \frac{r}{r-2M}\frac{\partial\,{}^{(1)}H_0^{\rm RW}}{\partial t} + \frac{r}{r-2M}\frac{\partial\,{}^{(1)}K^{\rm RW}}{\partial t} - 2\frac{M}{r(r-2M)}\,{}^{(1)}H_1^{\rm RW} \\
\frac{\partial^2\,{}^{(1)}K^{\rm RW}}{\partial r\partial t} &= \frac{1}{r}\frac{\partial\,{}^{(1)}H_0^{\rm RW}}{\partial t} + \frac{\ell(\ell+1)}{2r^2}\,{}^{(1)}H_1^{\rm RW} - \frac{r-3M}{r(r-2M)}\frac{\partial\,{}^{(1)}K^{\rm RW}}{\partial t} \\
\frac{\partial^2\,{}^{(1)}H_0^{\rm RW}}{\partial r\partial t} &= \frac{r}{r-2M}\frac{\partial^2\,{}^{(1)}H_1^{\rm RW}}{\partial t^2} + \frac{r-4M}{r(r-2M)}\frac{\partial\,{}^{(1)}H_0^{\rm RW}}{\partial t} \\
&\quad + \frac{3M-r}{r(r-2M)}\frac{\partial\,{}^{(1)}K^{\rm RW}}{\partial t} + \frac{\ell(\ell+1)}{2r^2}\,{}^{(1)}H_1^{\rm RW}\ .
\end{aligned}
\tag{62}
$$

We also obtain a second set of three equations containing second order derivatives with respect to r. This makes a total of six equations for the three unknown functions ${}^{(1)}H^{\rm RW}$, ${}^{(1)}H_1^{\rm RW}$, and ${}^{(1)}K^{\rm RW}$, where ${}^{(1)}H^{\rm RW} \equiv {}^{(1)}H_2^{\rm RW} = {}^{(1)}H_0^{\rm RW}$. Compatibility of the system requires then that these equations are not independent. One can show that by replacing the first three equations in the second set, one obtains a single compatibility condition (named the "algebraic identity" in Ref. [2]). This can be written as

$$
\begin{aligned}
&-\frac{\ell(\ell+1)M}{r^2}\,{}^{(1)}H_1^{\rm RW} - 2r\frac{\partial^2\,{}^{(1)}H_1^{\rm RW}}{\partial t^2} - \left[(\ell-1)(\ell+1) + \frac{6M}{r}\right]\frac{\partial\,{}^{(1)}H^{\rm RW}}{\partial t} \\
&+ \left[(\ell-1)(\ell+2) + \frac{2M(r-3M)}{r(r-2M)}\right]\frac{\partial\,{}^{(1)}K^{\rm RW}}{\partial t} + \frac{2r^3}{r-2M}\frac{\partial^3\,{}^{(1)}K^{\rm RW}}{\partial t^3} = 0\ .
\end{aligned}
\tag{63}
$$

This equation together with (62), provides a set of four differential equations for ${}^{(1)}H^{\rm RW}$, ${}^{(1)}H_1^{\rm RW}$, and ${}^{(1)}K^{\rm RW}$. One can again show that if (63) and two of the equations (62) are satisfied, then the remaining equation is also satisfied. We can now take the t derivative of the equations in (62) and use (63) to eliminate ${}^{(1)}H^{\rm RW}{}_{,t}$ in the second and third of the equations in (62). This reduces the system to two coupled linear partial differential equations for ${}^{(1)}H_1^{\rm RW}$ and ${}^{(1)}K^{\rm RW}$, with r-dependent coefficients, of first order in r and second order in t. As shown in Zerilli's paper this system can be "diagonalized" by the transformation,

$$
\partial\,{}^{(1)}K^{\rm RW}/\partial t = f(r)\,{}^{(1)}\chi + g(r)\,{}^{(1)}\widehat{R} \tag{64}
$$

$$
{}^{(1)}H_1^{\rm RW} = h(r)\,{}^{(1)}\chi + k(r)\,{}^{(1)}\widehat{R}\ , \tag{65}
$$

where

$$
\begin{aligned}
f(r) &= \frac{\lambda(\lambda+1)r^2 + 3\lambda Mr + 6M^2}{r^2(\lambda r - 3M)}\ ,\quad g(r) = 1 \\
h(r) &= \frac{\lambda r^2 - 3\lambda Mr - 3M^2}{(r-2M)(\lambda r + 3M)}\ ,\quad k(r) = \frac{r^2}{r-2M}\ ,\quad \lambda = \frac{(\ell-1)(\ell+2)}{2}.
\end{aligned}
\tag{66}
$$

A necessary condition for the Einstein equations to be satisfied is that ${}^{(1)}\chi$ and ${}^{(1)}\widehat{R}$ satisfy

$$
\frac{\partial\,{}^{(1)}\chi}{\partial r^*} = {}^{(1)}\widehat{R}\ ,\quad \frac{\partial\,{}^{(1)}\widehat{R}}{\partial r^*} = \left[V(r^*) + \frac{\partial^2}{\partial t^2}\right]\,{}^{(1)}\chi \tag{67}
$$

where

$$
V(r^*) = 2\left(1 - \frac{2M}{r}\right)\frac{\lambda^2 r^2\left[(\lambda+1)r + 3M\right] + 9M^2(\lambda r + M)}{r^3(\lambda r + 3M)^2}\ , \tag{68}
$$

and, as in (59),

$$r^* = r + 2M \ln[r/(2M) - 1] \ . \tag{69}$$

Equations (67) imply that ${}^{(1)}\chi$ satisfies the "Zerilli equation"

$$\frac{\partial^2\ {}^{(1)}\chi}{\partial r^{*2}} - \frac{\partial^2\ {}^{(1)}\chi}{\partial t^2} - V(r^*)\ {}^{(1)}\chi = 0 \ . \tag{70}$$

A necessary condition for perturbations ${}^{(1)}H_1^{RW}$ and ${}^{(1)}K^{RW}$ to satisfy Einstein's equations, is that they satisfy the relations in the system of equations (64)–(67), for ${}^{(1)}\chi$ satisfying (70). But in arriving at this system we have differentiated Einstein's equations with respect to t. It will be useful in what follows to set down here two of the original (not time differentiated) Einstein equations:

$$r\frac{\partial\ {}^{(1)}H_1^{RW}}{\partial t} = \frac{2M}{r}\ {}^{(1)}H^{RW} + (r-2M)\left(\frac{\partial\ {}^{(1)}H^{RW}}{\partial r} - \frac{\partial\ {}^{(1)}K^{RW}}{\partial r}\right) \tag{71}$$

$$\frac{r^3}{r-2M}\frac{\partial^2\ {}^{(1)}K^{RW}}{\partial t^2} = r(r-2M)\frac{\partial^2\ {}^{(1)}K^{RW}}{\partial r^2} + 2(2r-3M)\frac{\partial\ {}^{(1)}K^{RW}}{\partial r}$$

$$-\,2(r-2M)\frac{\partial\ {}^{(1)}H^{RW}}{\partial r} + 2r\frac{\partial\ {}^{(1)}H_1^{RW}}{\partial t} + (\ell-1)(\ell+2)\ {}^{(1)}K^{RW} - 2\ {}^{(1)}H^{RW} \ . \tag{72}$$

Once (70) is solved, the values of all the RW perturbations, can be found. One finds ${}^{(1)}\widehat{R}$ from the first of the relations in (67), and then from (64) and (65) one finds ${}^{(1)}H_1^{\rm RW}$ and ${}^{(1)}\dot{K}^{\rm RW}$. The solution for ${}^{(1)}\dot{H}^{\rm RW}$ is then found from the second of the equations in (62). Lastly the "integration constants" in finding ${}^{(1)}K^{\rm RW}$, ${}^{(1)}H^{\rm RW}$ from ${}^{(1)}\dot{K}^{\rm RW}$, ${}^{(1)}\dot{H}^{\rm RW}$ are known from the metric on the $t=0$ hypersurface, which fixes the initial values of ${}^{(1)}K^{\rm RW}$, ${}^{(1)}H^{\rm RW}$.

The Zerilli equation requires Cauchy data, values of ${}^{(1)}\chi$ and of $\partial\ {}^{(1)}\chi/\partial t$, at $t=0$, and the Cauchy data must originate in an initial value solution of Einstein's equations, i.e., our solution, in some arbitrary gauge, for the 3-metric γ_{ij} and the extrinsic curvature K_{ij}. From these we immediately get the $t=0$ values of ${}^{(1)}\widetilde{H}_2$, ${}^{(1)}\widetilde{h}_1$, ${}^{(1)}\widetilde{K}$ and ${}^{(1)}\widetilde{G}$. To proceed further it is convenient, though not necessary, to choose to view our initial data to be in a gauge with ${}^{(1)}\widetilde{h}_0 = {}^{(1)}\widetilde{H}_0 = {}^{(1)}\widetilde{H}_1 = 0$. In this gauge (61) gives us the time derivatives of ${}^{(1)}\widetilde{H}_2$, ${}^{(1)}\widetilde{h}_1$, ${}^{(1)}\widetilde{K}$, ${}^{(1)}\widetilde{G}$. immediately from K_{ij}. Using this information about the initial 3-geometry perturbations, and the definitions (44)–(46), we can find the initial values of ${}^{(1)}H^{\rm RW}$, ${}^{(1)}H_1^{\rm RW}$, ${}^{(1)}K^{\rm RW}$ and of the time derivatives ${}^{(1)}\dot{H}^{\rm RW}$, ${}^{(1)}\dot{K}^{\rm RW}$. As an example, from the Misner initial value solution, which has vanishing initial extrinsic curvature, if we choose ${}^{(1)}\widetilde{h}_0 = {}^{(1)}\widetilde{H}_0 = {}^{(1)}\widetilde{H}_1 = 0$, we have

$${}^{(1)}\widetilde{H}_2 = {}^{(1)}\widetilde{K} = 2\kappa_2\left(1-\frac{2M}{R}\right)^{-1}\left(\frac{2M}{R}\right)^3\sqrt{\frac{4\pi}{5}}$$

$${}^{(1)}\widetilde{G} = {}^{(1)}\widetilde{h}_1 = {}^{(1)}\widetilde{H}_{2,t} = {}^{(1)}\widetilde{K}_{,t} = {}^{(1)}\widetilde{G}_{,t} = {}^{(1)}\widetilde{h}_{1,t} = 0 \ . \tag{73}$$

With these we can solve (64) and (65) for ${}^{(1)}\chi$. It remains to find ${}^{(1)}\chi_{,t}$, which requires ${}^{(1)}H_1^{\rm RW}{}_{,t}$ and ${}^{(1)}K^{\rm RW}{}_{,tt}$. With the values of the RW perturbations, and derivatives, already found, ${}^{(1)}H_1^{\rm RW}{}_{,t}$ follows from (71), and ${}^{(1)}K^{\rm RW}{}_{,tt}$ from (72).

The above development of first order perturbation theory gives the time-domain Zerilli formalism in terms of the field variable ${}^{(1)}\chi$, defined [see (64) — (67)] by

$${}^{(1)}\chi = \frac{r-2M}{\lambda r + 3M}\left[\frac{r^2}{r-2M}\frac{\partial\ {}^{(1)}K^{\rm RW}}{\partial t} - {}^{(1)}H_1^{\rm RW}\right] \ . \tag{74}$$

This development is important here because it will be used as a pattern for the second order formalism. For the first order computations themselves, however, it is convenient not actually to use ${}^{(1)}\chi$, but rather to use a field variable ${}^{(1)}\psi$ that is roughly equivalent to the time integral of ${}^{(1)}\chi$.

To introduce ${}^{(1)}\psi$ we start by using the second relationship in (62) to replace ${}^{(1)}H_1^{\rm RW}$ in (74), with the result

$$ {}^{(1)}\chi = \frac{2r(r-2M)}{\ell(\ell+1)(\lambda r+3M)}\left[{}^{(1)}\dot{H}^{\rm RW} - r\frac{\partial\ {}^{(1)}\dot{K}^{\rm RW}}{\partial r} - \frac{r-3M}{r-2M}\ {}^{(1)}\dot{K}^{\rm RW}\right] + \frac{r^2}{\lambda r+3M}\ {}^{(1)}\dot{K}^{\rm RW}\ . \tag{75} $$

This leads us to define

$$ {}^{(1)}\psi \equiv \frac{2r(r-2M)}{\ell(\ell+1)(\lambda r+3M)}\left[{}^{(1)}H^{\rm RW} - r\frac{\partial\ {}^{(1)}K^{\rm RW}}{\partial r} - \frac{r-3M}{r-2M}\ {}^{(1)}K^{\rm RW}\right] + \frac{r^2}{\lambda r+3M}\ {}^{(1)}K^{\rm RW}\ , \tag{76} $$

which, from (75) satisfies

$$ {}^{(1)}\dot{\psi} = {}^{(1)}\chi\ . \tag{77} $$

It is straightforward to show that ${}^{(1)}\psi$, like ${}^{(1)}\chi$, satisfies the Zerilli equation

$$ \frac{\partial^2\ {}^{(1)}\psi}{\partial r^{*2}} - \frac{\partial^2\ {}^{(1)}\psi}{\partial t^2} - V(r^*)\ {}^{(1)}\psi = 0\ . \tag{78} $$

It will be useful for the analysis below of asymptotic behavior to note that in terms of ${}^{(1)}\psi$ the equivalents of (64), (65), and (67) can be written

$$ {}^{(1)}K^{RW} = f(r)\ {}^{(1)}\psi + \left(1-\frac{2M}{r}\right)\frac{\partial\ {}^{(1)}\psi}{\partial r} \tag{79} $$

$$ {}^{(1)}H_1^{RW} = h(r)\frac{\partial\ {}^{(1)}\psi}{\partial t} + r\frac{\partial^2\ {}^{(1)}\psi}{\partial t\partial r} \tag{80} $$

$$ {}^{(1)}H^{RW} = \frac{\partial}{\partial r}\left[\left(1-\frac{2M}{r}\right)h(r)\ {}^{(1)}\psi + r\frac{\partial\ {}^{(1)}\psi}{\partial r}\right] - {}^{(1)}K^{RW}\ . \tag{81} $$

The wavefunction ${}^{(1)}\psi$ is very closely related to the variable used by Moncrief [28], which is defined in terms of perturbations in an arbitrary gauge to be

$$ {}^{(1)}\psi_{\rm Monc} \equiv \frac{2r(r-2M)}{\ell(\ell+1)(\lambda r+3M)}\left[{}^{(1)}\widetilde{H}_2 - r\frac{\partial\ {}^{(1)}\widetilde{K}}{\partial r} - \frac{r-3M}{r-2M}\ {}^{(1)}\widetilde{K}\right] $$

$$ + \frac{r^2}{\lambda r+3M}\left[{}^{(1)}\widetilde{K} + (r-2M)\left(\frac{\partial\ {}^{(1)}\widetilde{G}}{\partial r} - \frac{2}{r^2}\ {}^{(1)}\widetilde{h}_1\right)\right]. \tag{82} $$

(Note that our expression for ${}^{(1)}\psi_{\rm Monc}$ must be multiplied by $\ell(\ell+1)$ to get the expression in Moncrief's paper.) It is clear that ${}^{(1)}\psi_{\rm Monc}$ reduces to ${}^{(1)}\psi$, when one specializes to the RW gauge and with the use of one of the vacuum field equations sets ${}^{(1)}H_2^{\rm RW} = {}^{(1)}H^{\rm RW}$. The great advantage of Moncrief's formulation in (82) is that ${}^{(1)}\psi_{\rm Monc}$ is defined entirely in terms of components of the 3-geometry. This greatly simplifies finding Cauchy data from an initial value solution.

The simple relationship of the Zerilli and the Moncrief wavefunctions disappears in the presence of real source terms, or of the effective source terms that appear in the second order equations. In our second order development we could have followed either the pattern leading to a Zerilli equation for a second order variable analogous to ${}^{(1)}\chi$, or we could have followed the pattern of Moncrief's derivation [28] to find a Zerilli equation for a variable analogous to ${}^{(1)}\psi_{\rm Monc}$. The former choice has the disadvantage that it is related to the metric perturbations by an additional time derivative, but it has the advantage of greater simplicity than the Moncrief approach. The complexity of the second order equations makes simplicity the more important consideration, and we have chosen to work with a second order equivalent of ${}^{(1)}\chi$.

B. Radiation

The solution for first order perturbations does not directly give us the flux of radiated power. To analyze the radiation we must go to a coordinate system in which the metric has a manifestly asymptotically flat (AF) form, in which the deviations from the Schwarzschild metric fall off with radius as follows.(See Ref. [22], Chap. 19.)

$$\delta g_{00}, \delta g_{01}, \delta g_{11}, = \mathcal{O}(r^{-2}) \quad \delta g_{02}, \delta g_{03}, \delta g_{12}, \delta g_{13} = \mathcal{O}(r^{-1})$$

$$\delta g_{22}, \delta g_{23}, \delta g_{33} = \mathcal{O}(r) \ . \tag{83}$$

In this coordinate system the information about gravitational radiation is carried by the transverse metric components $\delta g_{22}, \delta g_{23}, \delta g_{33}$, and these perturbations will have the form $r\times$function of $t - r^*$.

These are requirements on the r dependence of the metric functions for the coordinate system to be AF. If it is to be AF to first order then these conditions must be satisfied by the metric perturbations to first order. The RW gauge is not AF. There are two rather different ways in which we can extract radiation information from our RW results. The first method is to use the gauge invariant expressions for the first order ${}^{(1)}\chi$ and $Q^{\rm odd}$, and to look at the form they take in the radiation zone, for an asymptotically flat gauge. To first order, the AF gauge conditions requires that AF perturbations fall of as

$$H_0^{\rm AF}, H_1^{\rm AF}, H_2^{\rm AF} \sim r^{-2} \quad h_0^{\rm AF}, h_1^{\rm AF}, G^{\rm AF}, K^{\rm AF} \sim r^{-1} \tag{84}$$

$$h_0^{\rm AF,odd}, h_1^{\rm AF,odd} \sim r^{-1} \quad h_1^{\rm AF,odd} \sim r \tag{85}$$

For first order odd parity perturbations, (42), (58) in the radiation zone, then give us,

$$Q^{\rm odd} \approx r^{-1}\ {}^{(1)}h_1^{\rm RW,odd} \approx -\tfrac{1}{2} r^{-1}\ {}^{(1)}\dot{h}_2^{\rm AF,odd} \ , \tag{86}$$

where the dot means differentiation with respect to time, or to retarded time. Of particular interest is the power radiated. From the asymptotic form of the perturbations, one finds [9]

$$\text{Power} = \epsilon^2 \frac{1}{16\pi} \frac{(\ell+2)!}{(\ell-2)!} \left(Q^{\rm odd}\right)^2 \ . \tag{87}$$

For even parity, in the radiation zone, using (74), the relation equivalent to (86) is:

$${}^{(1)}\chi \approx \tfrac{1}{\lambda} \left(r\ {}^{(1)}\dot{K}^{\rm RW} - {}^{(1)}H_1^{\rm RW} \right) \ ,$$

or, using the first order gauge transformation equations,

$${}^{(1)}\chi \approx \frac{r}{\lambda} \left({}^{(1)}\dot{K}^{\rm AF} - {}^{(1)}\dot{G}^{\rm AF} \right) \ , \tag{88}$$

and from this asymptotic form, one finds that the radiated power is

$$\text{Power} = \epsilon^2 \frac{\pi}{4} \frac{(\ell+2)!}{(\ell-2)!} \frac{1}{(2\ell+1)^2} \chi^2 \ . \tag{89}$$

A very different procedure for extracting radiation is to perform an explicit gauge transformation from the RW gauge to a gauge that is AF. Since this will be the basis of the approach we use for the second order problem, we illustrate it here. For simplicity of presentation, we limit attention to the even-parity $\ell = 2$ case, the case of interest for second order calculations. We shall also write the perturbative functions in terms of ${}^{(1)}\psi(t,r)$ instead of ${}^{(1)}\chi(t,r)$ to avoid having to deal with ${}^{(1)}K^{\rm RW}{}_{,t}$, and ${}^{(1)}H^{\rm RW}{}_{,t}$, instead of ${}^{(1)}K^{\rm RW}$, and ${}^{(1)}H^{\rm RW}$ We start by noticing that the solution to the Zerilli equation (78), for outgoing radiation, can be written in the form of an expansion in powers of r in which the coefficients are functions of retarded time $t - r^*$. That is, we write ${}^{(1)}\psi(t,r)$ in the form

$${}^{(1)}\psi(t,r) = {}^{(1)}F_a(t-r^*) + {}^{(1)}F_b(t-r^*)/r + {}^{(1)}F_c(t-r^*)/r^2 + \mathcal{O}(1/r^3) \ . \tag{90}$$

Substituting this *ansatz* in (78) one can show that ${}^{(1)}F_b$, ${}^{(1)}F_c, \cdots$ are determined once, e.g., ${}^{(1)}F_a$ is fixed. For the first terms in the expansion (90), these relations may be conveniently expressed in terms of a suitably defined function $F(t)$ as,

$${}^{(1)}F_a(t) = \frac{1}{\lambda+1}\ {}^{(1)}F''(t) \quad , \quad {}^{(1)}F_b(t) = {}^{(1)}F'(t) \quad , \quad {}^{(1)}F_c(t) = \frac{\lambda}{2}\ {}^{(1)}F(t) - \frac{3M(\lambda+2)}{2\lambda(\lambda+1)}\ {}^{(1)}F'(t) \ , \tag{91}$$

where $F'(x) = dF(x)/dx$, $F''(x) = d^2F(x)/dx^2$, etc. In particular, for the quadrupole contribution $\ell = 2$, we have,

$$ {}^{(1)}\psi = \frac{1}{3}\ {}^{(1)}F''(t-r^*) + \frac{1}{r}\ {}^{(1)}F'(t-r^*) + \frac{1}{r^2}\left[\ {}^{(1)}F - M\ {}^{(1)}F'\right] + \cdots \ . \tag{92} $$

The form of ${}^{(1)}\psi$ and hence of ${}^{(1)}F(t)$ will, of course, be determined by the Cauchy data used in the solution of (78).

For $r \gg M$ we may obtain the asymptotic RW perturbation functions in terms of ${}^{(1)}\psi$, by replacing in (79), (80), (81) the asymptotic form of ${}^{(1)}\psi$. Restricting again for simplicity to $\ell = 2$, we find:

$$ \begin{aligned} K^{\rm RW} &= \frac{\partial\ {}^{(1)}\psi}{\partial r} + \frac{3}{r}\ {}^{(1)}\psi - \frac{2M}{r}\frac{\partial\ {}^{(1)}\psi}{\partial r} + O(1/r^2) \\ H_0^{\rm RW} &= r\frac{\partial^2\ {}^{(1)}\psi}{\partial r^2} - 2M\frac{\partial^2\ {}^{(1)}\psi}{\partial r^2} + \frac{\partial\ {}^{(1)}\psi}{\partial r} - \frac{3}{r}\ {}^{(1)}\psi - \frac{5M}{2r}\frac{\partial\ {}^{(1)}\psi}{\partial r} + O(1/r^2) \\ H_1^{\rm RW} &= r\frac{\partial^2\ {}^{(1)}\psi}{\partial r\partial t} + \frac{\partial\ {}^{(1)}\psi}{\partial t} - \frac{5M}{2r}\frac{\partial\ {}^{(1)}\psi}{\partial t} + O(1/r^2) \\ H_2^{\rm RW} &= H_0^{\rm RW}\ . \end{aligned} \tag{93} $$

From these relations, and from (92), we see that the RW perturbations diverge for $r \to \infty$. We can view these divergent perturbations as an indication that the RW gauge is not asymptotically flat. (Here we are considering the RW quantities as perturbations in a specific gauge, the RW gauge; we usually view them as gauge invariant combinations of perturbations). We can explicitly perform a gauge transformation, analogous to that in (37), to take the perturbations from RW gauge to AF gauge:

$$ \begin{aligned} {}^{(1)}H_0^{\rm AF} &= {}^{(1)}H_0^{\rm RW} + \frac{2M}{r(r-2M)}\ {}^{(1)}\alpha_1 + 2\frac{\partial\ {}^{(1)}\alpha_0}{\partial t} \\ {}^{(1)}H_1^{\rm AF} &= {}^{(1)}H_1^{\rm RW} - \frac{r}{r-2M}\frac{\partial\ {}^{(1)}\alpha_1}{\partial t} + \frac{r-2M}{r}\frac{\partial\ {}^{(1)}\alpha_0}{\partial r} \\ {}^{(1)}h_0^{\rm AF} &= -r^2\frac{\partial\ {}^{(1)}\alpha_2}{\partial t} + \frac{r-2M}{r}\ {}^{(1)}\alpha_0 \\ {}^{(1)}H_2^{\rm AF} &= {}^{(1)}H_2^{\rm RW} + \frac{2M}{r(r-2M)}\ {}^{(1)}\alpha_1 - 2\frac{\partial\ {}^{(1)}\alpha_1}{\partial r} \\ {}^{(1)}h_1^{\rm AF} &= -\frac{r}{r-2M}\ {}^{(1)}\alpha_1 - r^2\frac{\partial\ {}^{(1)}\alpha_2}{\partial r} \\ {}^{(1)}G^{\rm AF} &= -2\ {}^{(1)}\alpha_2 \\ {}^{(1)}K^{\rm AF} &= {}^{(1)}K^{\rm RW} - \frac{2}{r}\ {}^{(1)}\alpha_1\ , \end{aligned} \tag{94} $$

where we are denoting with ${}^{(1)}\alpha_0$, ${}^{(1)}\alpha_1$, ${}^{(1)}\alpha_2$, the specification of gauge vector ${}^{(1)}\xi$, the role played by ${}^{(1)}A_0$, ${}^{(1)}A_1$, ${}^{(1)}A_2$, in (37). To cancel the leading term in ${}^{(1)}K^{\rm RW}$, and force the appropriate asymptotic behavior for $K^{\rm AF}$ we choose

$$ {}^{(1)}\alpha_1 = \frac{r}{2}\frac{\partial\ {}^{(1)}\psi}{\partial r}\ . \tag{95} $$

Similarly, the leading divergences in ${}^{(1)}H_0^{\rm RW}$ and ${}^{(1)}H_2^{\rm RW}$ are cancelled by the choice

$$ {}^{(1)}\alpha_0 = -\frac{r}{2}\frac{\partial\ {}^{(1)}\psi}{\partial t} \tag{96} $$

Finally, to avoid a "bad" behavior in ${}^{(1)}h_0$ and ${}^{(1)}h_1$ we choose,

$$ {}^{(1)}\alpha_2 = -\frac{1}{2r}\ {}^{(1)}\psi\ . \tag{97} $$

This procedure may be iterated to the desired degree of accuracy. Since we are only interested in showing that there is a choice of gauge functions that carries the RW gauge to an asymptotically flat gauge, the computations are simplified if we consider from the start the expression for the RW gauge perturbations in terms of ${}^{(1)}F$. To put the asymptotic form of the full metric in an "asymptotically flat" gauge, we need an expansion of ${}^{(1)}\chi$ up to terms of order $1/r^3$. The corresponding asymptotic forms for ${}^{(1)}H^{\rm RW}$, ${}^{(1)}H_1^{\rm RW}$, and ${}^{(1)}K^{\rm RW}$ are then,

$$
\begin{aligned}
{}^{(1)}H^{\rm RW} &= \frac{r}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) + \frac{2M}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) + \frac{2}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) \\
&\quad + \frac{1}{r}\left[\frac{4M^2}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) + \frac{11M}{6}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) + \frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*)\right] + {\rm O}(1/r^2) \\
{}^{(1)}H_1^{\rm RW} &= -\frac{r}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) - \frac{2M}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) - \frac{2}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) \\
&\quad - \frac{1}{r}\left[\frac{4M^2}{3}\frac{\partial^4\,{}^{(1)}F}{\partial t^4}(t-r^*) + \frac{11M}{6}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) + \frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*)\right] + {\rm O}(1/r^2) \\
{}^{(1)}K^{\rm RW} &= -\frac{1}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) + \frac{1}{r^2}\left[\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*) + \frac{M}{2}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*)\right] \\
&\quad + \frac{1}{r^3}\left[\,{}^{(1)}F(t-r^*) + \frac{M}{2}\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*)\right] + {\rm O}(1/r^2)\ .
\end{aligned}
\tag{98}
$$

By using these expansions on the right hand side of (94) we then find for ${}^{(1)}\alpha_0$, ${}^{(1)}\alpha_1$ and ${}^{(1)}\alpha_2$ the asymptotic expansions,

$$
\begin{aligned}
{}^{(1)}\alpha_0 &= -\frac{r}{6}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) - \frac{M}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) - \frac{1}{3}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*) \\
&\quad + \frac{1}{r}\left[-\frac{1}{2}\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*) - \frac{3M}{4}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*) - \frac{2M^2}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*)\right] \\
&\quad + \frac{1}{r^2}\left[-\frac{1}{2}\,{}^{(1)}F(t-r^*) - \frac{3M}{4}\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*) - \frac{3M^2}{2}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*)\right. \\
&\quad \left. - \frac{4M^3}{3}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*)\right] + \frac{1}{r^3}\mathcal{A}_0(t-r^*) + {\rm O}(1/r^4) \\
{}^{(1)}\alpha_1 &= -\frac{r}{6}\frac{\partial^3\,{}^{(1)}F}{\partial t^3}(t-r^*) - \frac{1}{2}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*) + \frac{1}{r}\left[-\frac{1}{2}\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*) + \frac{M}{4}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*)\right] \\
&\quad + \frac{1}{r^3}\mathcal{A}_1(t-r^*) + {\rm O}(1/r^4) \\
{}^{(1)}\alpha_2 &= -\frac{1}{6r}\frac{\partial^2\,{}^{(1)}F}{\partial t^2}(t-r^*) - \frac{1}{3r^2}\frac{\partial\,{}^{(1)}F}{\partial t}(t-r^*) + \frac{1}{r^3}\mathcal{A}_2(t-r^*) + +{\rm O}(1/r^4)
\end{aligned}
\tag{99}
$$

With these expansions, and (98) in (94), we finally find

$$
\begin{aligned}
{}^{(1)}H_0^{\rm AF} &= \frac{1}{r^3}\left[\frac{16}{3}M^4\,{}^{(1)}F''''(t-r^*) + \frac{69}{8}M^3\,{}^{(1)}F'''(t-r^*) + \frac{5}{4}M^2\,{}^{(1)}F''(t-r^*)\right. \\
&\quad \left. + \,{}^{(1)}F(t-r^*) + 2\mathcal{A}_0{}'(t-r^*) + \mathcal{F}''(t-r^*)\right] + {\rm O}(1/r^4) \\
{}^{(1)}H_1^{\rm AF} &= \frac{1}{r^3}\left[-\frac{8}{3}M^4\,{}^{(1)}F''''(t-r^*) - \frac{45}{8}M^3\,{}^{(1)}F'''(t-r^*) + \frac{3}{4}M^2\,{}^{(1)}F''(t-r^*)\right. \\
&\quad \left. + \,{}^{(1)}F(t-r^*) - \mathcal{A}_0{}'(t-r^*) - \mathcal{A}_1{}'(t-r^*) - \mathcal{F}''(t-r^*)\right] + {\rm O}(1/r^4) \\
{}^{(1)}H_2^{\rm AF} &= \frac{1}{r^3}\left[-\frac{21}{8}M^3\,{}^{(1)}F'''(t-r^*) - \frac{11}{4}M^2\,{}^{(1)}F''(t-r^*)\right. \\
&\quad \left. + \,{}^{(1)}F(t-r^*) + 2\mathcal{A}_1'(t-r^*) + \mathcal{F}''(t-r^*)\right] + {\rm O}(1/r^4) \\
{}^{(1)}h_0^{\rm AF} &= \frac{1}{r}\left[-\frac{M}{12}\,{}^{(1)}F'''(t-r^*) - \frac{1}{2}\,{}^{(1)}F'(t-r^*) - \mathcal{A}_2'(t-r^*)\right] + {\rm O}(1/r^2) \\
{}^{(1)}h_1^{\rm AF} &= \frac{1}{r}\left[\frac{M}{12}\,{}^{(1)}F'''(t-r^*) - \frac{1}{6}\,{}^{(1)}F'(t-r^*) + \mathcal{A}_2'(t-r^*)\right] + {\rm O}(1/r^2) \\
{}^{(1)}G^{\rm AF} &= \frac{1}{3r}\,{}^{(1)}F''(t-r^*) + \frac{2}{3r^2}\,{}^{(1)}F'(t-r^*) - \frac{2}{r^3}\mathcal{A}_2(t-r^*) + {\rm O}(1/r^4) \\
{}^{(1)}K^{\rm AF} &= \frac{1}{r}\,{}^{(1)}F''(t-r^*) + \frac{2}{r^2}\,{}^{(1)}F'(t-r^*) + \frac{1}{2r^3}\left[M\,{}^{(1)}F'(t-r^*) + 2\,{}^{(1)}F(t-r^*)\right] + {\rm O}(1/r^4)
\end{aligned}
\tag{100}
$$

In all these expressions $\mathcal{F}$ is the contribution of order $1/r^4$ in ${}^{(1)}\chi$, and we notice that $\mathcal{A}_0$,$\mathcal{A}_1$ and $\mathcal{A}_2$ are not determined by the condition of asymptotic flatness, and can be freely specified.

For computational purposes, it is useful to have expressions for the asymptotic forms for ${}^{(1)}\alpha_0$, ${}^{(1)}\alpha_1$ and ${}^{(1)}\alpha_2$ in terms of ${}^{(1)}\psi$, instead of ${}^{(1)}F$ as in (100). This inversion is rather cumbersome, because a given term in ${}^{(1)}\psi$ contributes to many orders. The leading behavior found in (100) is reproduced by,

$$
\begin{aligned}
{}^{(1)}\alpha_2 &= -\frac{1}{2r}\,{}^{(1)}\psi + \frac{1}{2r^2}\int {}^{(1)}\psi dt \\
{}^{(1)}\alpha_0 &= -\frac{r}{2}\frac{\partial\,{}^{(1)}\psi}{\partial t} + \frac{1}{2}\,{}^{(1)}\psi - M\frac{\partial\,{}^{(1)}\psi}{\partial t} \\
{}^{(1)}\alpha_1 &= \frac{r}{2}\frac{\partial\,{}^{(1)}\psi}{\partial r} + M\frac{\partial\,{}^{(1)}\psi}{\partial t}
\end{aligned}
\tag{101}
$$

but more terms need to be specified to achieve the same order of accuracy as in (100). In practice one needs only the leading terms in (100), and having solved for ${}^{(1)}\psi$, one can find ${}^{(1)}F$, to needed accuracy, from the $\mathcal{O}(r^0)$ and $\mathcal{O}(r^{-1})$ parts of ${}^{(1)}\psi$.

We have shown that there are gauge transformations such that the metric functions can be put in the asymptotic form, and in which the gravitational wave amplitudes (the $\mathcal{O}(r^0)$ terms in ${}^{(1)}F^{\rm AF}$ and ${}^{(1)}G^{\rm AF}$) can be found from ${}^{(1)}\psi$, or, up to an irrelevant, r dependent 'integration constant', in terms of ${}^{(1)}\chi$ In the next section we explore the possibility of obtaining similar results in the second order perturbations.

IV. SECOND ORDER PERTURBATIONS

A. Wave Equations

We now turn to the Einstein equations at second order in the perturbations, and find the first order formalism of the preceding section can be modified to handle higher orders. In the second order Einstein equations (26),

$$
L_{\lambda\tau}({}^{(2)}g_{\alpha\beta}) = {}^{(2)}\mathcal{T}_{\lambda\tau}\left({}^{(1)}g_{\alpha\beta}\right) \ , \tag{102}
$$

the left hand side of the second order equations is identical in form to that of the first order equation. The only difference is that the perturbations with index $n = 1$ of first order theory are replaced by those with index $n = 2$. Since the $L_{\lambda\tau}$ operator is spherically symmetric for a Schwarzschild background, and linear, $L_{\lambda\tau}({}^{(2)}g_{\alpha\beta})$ can be decomposed into multipoles, and the ℓm multipole of $L_{\lambda\tau}$ will involve only that part of ${}^{(2)}g_{\alpha\beta}$ with the same ℓm, and the same parity.

The right hand side of (102) can be expanded, as can any functions of (θ, ϕ), in even and odd multipoles so, for example, we can write:

$$
\begin{aligned}
{}^{(2)}\mathcal{T}_{00} &= \sum_{\ell,m} {}^{(2)}\mathcal{A}^{(\ell,m)} Y^\ell_m \\
{}^{(2)}\mathcal{T}_{02} &= \sum_{\ell,m} {}^{(2)}\mathcal{B}^{(\ell,m)}\frac{\partial Y^\ell_m}{\partial\theta} + {}^{(2)}\mathcal{B}^{(\ell,m)\rm odd}\frac{1}{\sin\theta}\frac{\partial Y^\ell_m}{\partial\phi} \ ,
\end{aligned}
\tag{103}
$$

and so forth. The multipoles of ${}^{(2)}\mathcal{T}_{00}$, for example, are found from

$$
{}^{(2)}\mathcal{A}^{(\ell,m)} = \int {}^{(2)}\mathcal{T}_{00}\left(Y^\ell_m\right)^* d\cos\theta d\phi \ . \tag{104}
$$

It is important to understand that such a multipole decomposition can always be made, but particular ℓm multipole of ${}^{(2)}\mathcal{T}_{\lambda\tau}$ will not be directly related to the same ℓm multipole of the first order perturbations ${}^{(1)}g_{\alpha\beta}$. The nonlinear dependence of ${}^{(2)}\mathcal{T}_{\lambda\tau}$ on the first order perturbations will mix multipoles and parities. Thus, for example, the $\ell = 2$ second order perturbations will be driven by terms on the right hand side coming from the product of first order perturbations with $\ell = 2$ and those with $\ell = 4$. Even parity second order perturbations will be driven by the products of odd parity first order perturbations, etc.

As an example of this mixing, Cunningham *et al.* [9] considered the collapse of a rotating relativistic stellar model, to second order in the rate of rotation. The star's first order perturbation is the conserved $\ell = 1$, odd parity, angular momentum perturbation. To second order, this rotation drives an even parity $\ell = 2$ radiatable perturbation.

The detailed expressions for the projection of the multipole components of expressions that are quadratic in multipole expansions (as are the ${}^{(2)}\mathcal{T}_{\lambda\tau}$) is tedious, though straightforward [6], and will be omitted here. The manner in which it is carried out depends on the details of a perturbation problem. In some instances the multipole projections might best be carried out numerically. In what follows we shall suppose that the necessary multipole projections have been carried out.

We shall make additional assumptions; we shall consider only the even parity $\ell = 2$ second order multipole. One justification for this is that it will simplify rather lengthy expressions and greatly simplify their description. This will help the presentation to focus on important basic issues about higher order perturbation theory with minimal distraction from a minor complication. The generalization from $\ell = 2$ to any other multipole is absolutely straightforward. We also choose to focus on the quadrupole for a practical reason. Gravitational wave generation seem almost always to be dominated by quadrupole radiation, even for sources in which the usual arguments (slow-motion) for quadrupole dominance do not apply. The justification for the even parity analysis, is that it is a more difficult system to work with. By describing the even parity second order problem we believe we are laying the foundation for a reader to make a similar (but simpler) extension from first order to second order odd parity analysis.

Our next step for second order analysis is to perform a *purely second order* gauge transformation that sets ${}^{(2)}h_0 = {}^{(2)}h_1 = {}^{(2)}G = 0$. The second order gauge functions ${}^{(2)}\xi^\mu$, or equivalently ${}^{(2)}A_0$, ${}^{(2)}A_1$, ${}^{(2)}A_2$, needed are analogous to their first order counterparts, since all changes are taking place purely at the second order. Equations (33) – (47) hold true in the second order as well as first; only the indices "(1)" must be changed to "(2)".

At this point the range of gauge possibilities is wide, and potentially confusing. (i) We could start in some arbitrary gauge and transform *only* the second order perturbations to the RW gauge. That is we could choose to set ${}^{(2)}h_0 = {}^{(2)}h_1 = {}^{(2)}G = 0$, but not necessarily ${}^{(1)}h_0 = {}^{(1)}h_1 = {}^{(1)}G = 0$. We would then be in a second order, but not first order, RW gauge. (ii) We could start from an arbitrary gauge and use a first order transformation (carried out to at least second order, of course) to impose the conditions ${}^{(1)}h_0 = {}^{(1)}h_1 = {}^{(1)}G = 0$ but not necessarily the conditions ${}^{(2)}h_0 = {}^{(2)}h_1 = {}^{(2)}G = 0$. We would then be in a first order RW gauge, but not a second order RW gauge. (iii) We could choose to make a first order transformation (carried out at least to second order) to a first order RW gauge, followed by a purely second order transformation (which does not affect the first order gauge) to a second order RW gauge. We would then have $h_0 = h_1 = G = 0$ to both first and second order, and would be in a first and second order RW gauge.

It is easy to overlook some of the subtleties hidden in the nonlinear interactions of the first and second order gauge transformations. One should notice, for example, that the second order gauge functions ${}^{(2)}\xi^\mu$ used to set ${}^{(2)}h_0 = {}^{(2)}h_1 = {}^{(2)}G = 0$ will depend on whether we have *first* transformed to the first order RW gauge (because such a transformation changes the metric to second order). Since the non-vanishing second order RW parts ${}^{(2)}H_0^{\rm RW}$, ${}^{(2)}H_1^{\rm RW}$, ${}^{(2)}H_2^{\rm RW}$, ${}^{(2)}K^{\rm RW}$, depend on ${}^{(2)}\xi^\mu$, we should keep in mind that there are not *unique* second order RW perturbations. The second order RW perturbations, depend on the first order gauge.

In a second order RW gauge, the second order Einstein equations (102) consist of seven equations linear in the second order functions ${}^{(2)}H_1^{\rm RW}$, ${}^{(2)}H_0^{\rm RW}$, ${}^{(2)}H_2^{\rm RW}$, and ${}^{(2)}K^{\rm RW}$, but quadratic in the first order functions ${}^{(1)}H_1^{\rm RW}$, $(1)H_0^{\rm RW}$, and ${}^{(1)}K^{\rm RW}$. One of these equations is

$$
{}^{(2)}H_2^{\rm RW} = {}^{(2)}H_0^{\rm RW} + \mathcal{S}_{\rm diff} \tag{105}
$$

where $\mathcal{S}_{\rm diff}$ is quadratic in the first order perturbations. Since the first order problem is solved independently, $\mathcal{S}_{\rm diff}$ can be thought of as a known "source," and we shall use the term "source" to refer below to similar expressions quadratic in first order perturbations.

It is important to understand that the relation (105) exists in a second order RW gauge, whether or not we are using a *first* order RW gauge, but the source term $\mathcal{S}_{\rm diff}$ will be different (it will have a different numerical value at a given coordinate location t, r) depending on the gauge choice that has been made at first order. When it is important to emphasize the first order gauge choice that was made in computing source terms like $\mathcal{S}_{\rm diff}$, we will use a superscript to indicate the first order gauge, so that for example $\mathcal{S}_{\rm diff}^{\rm RW}$ indicates that a first order RW gauge was used, and $\mathcal{S}_{\rm diff}^{\rm AF}$ indicates a first order asymptotically flat gauge.

In whatever first order gauge, (105) is the second order equivalent of the first order relationship ${}^{(1)}H_2^{\rm RW} = {}^{(1)}H_0^{\rm RW}$, and can be used to eliminate ${}^{(2)}H_2^{\rm RW}$ in the remaining equations, as was done in the first order case. Note that unlike the first order case, we do not now introduce a symbol ${}^{(2)}H^{\rm RW}$ to represent both ${}^{(2)}H_2^{\rm RW}$ and ${}^{(2)}H_0^{\rm RW}$, since these second order quantities are not equal. As in the first order case, the Einstein equations consist of two sets of three equations for ${}^{(2)}H_1^{\rm RW}$, ${}^{(2)}H_0^{\rm RW}$, and ${}^{(2)}K^{\rm RW}$, one set containing only derivatives of first order in r, and the other with derivatives of second order in r. The functions ${}^{(2)}H_1^{\rm RW}$, ${}^{(2)}H_0^{\rm RW}$, and ${}^{(2)}K^{\rm RW}$ appear linearly in these

equations, which have the same form, with the same coefficients, as for the system for the corresponding first order functions ${}^{(1)}H_1^{\rm RW}$, $(1)H_0^{\rm RW}$, and ${}^{(1)}K^{\rm RW}$, but now with "source" terms quadratic in first order perturbations (and dependent on the first order gauge choice). The equations of first order in r derivatives are

$$\frac{\partial^2\ {}^{(2)}K^{\rm RW}}{\partial r\partial t} = \frac{1}{r}\frac{\partial\ {}^{(2)}H_0^{\rm RW}}{\partial t} + \frac{3}{r^2}\ {}^{(2)}H_1^{\rm RW} - \frac{r-3M}{r(r-2M)}\frac{\partial\ {}^{(2)}K^{\rm RW}}{\partial t} + \mathcal{S}_K \tag{106}$$

$$\begin{aligned}\frac{\partial^2\ {}^{(2)}H_0^{\rm RW}}{\partial r\partial t} &= \frac{r}{r-2M}\frac{\partial^2\ {}^{(2)}H_1^{\rm RW}}{\partial t^2} + \frac{r-4M}{r(r-2M)}\frac{\partial\ {}^{(2)}H_0^{\rm RW}}{\partial t}\\ &\quad + \frac{3M-r}{r(r-2M)}\frac{\partial\ {}^{(2)}K^{\rm RW}}{\partial t} + \frac{3}{r^2}\ {}^{(2)}H_1^{\rm RW} + \mathcal{S}_{H2}\end{aligned} \tag{107}$$

$$\begin{aligned}\frac{\partial\ {}^{(2)}H_1^{\rm RW}}{\partial r} &= \frac{r}{r-2M}\frac{\partial\ {}^{(2)}H_0^{\rm RW}}{\partial t} + \frac{r}{r-2M}\frac{\partial\ {}^{(2)}K^{\rm RW}}{\partial t}\\ &\quad -2\frac{M}{r(r-2M)}\ {}^{(2)}H_1^{\rm RW} + \mathcal{S}_{H1}\ ,\end{aligned} \tag{108}$$

where $\mathcal{S}_K$ and $\mathcal{S}_{H2}$ are "source terms," quadratic in first order perturbations, and in the r and t derivatives of first order perturbations.

As in the first order case, the second order RW perturbations must satisfy the remaining Einstein equations, three equations with second order derivatives in r. These equations are analogous to the corresponding first order equations, but the second order equations contain source terms quadratic in the first order perturbations. The procedure that was used to simplify the first order system works also in the second order case. For each arrangement of substitutions a different source term appears, but one can show that the source terms are equal if the first order perturbations that appear in the source terms satisfy the first order perturbation equations. This means that the equations that must be solved are three equations of first order in r derivatives, plus an "algebraic identity," of the form

$$\begin{aligned}&\frac{r^2}{r-2M}\frac{\partial^3\ {}^{(2)}K^{\rm RW}}{\partial t^3} - \frac{\partial^2\ {}^{(2)}H_1^{\rm RW}}{\partial t^2} + \frac{2r^2-3Mr-3M^2}{r^2(r-2M)}\frac{\partial\ {}^{(2)}K^{\rm RW}}{\partial t} - \frac{3M}{r^3}\ {}^{(2)}H_1^{\rm RW}\\ &\quad -\frac{2r+3M}{r^2}\frac{\partial\ {}^{(2)}H_0^{\rm RW}}{\partial t} + \mathcal{S}_{AI} = 0\ ,\end{aligned} \tag{109}$$

where $\mathcal{S}_{AI}$ is a "source" term, quadratic in the first order perturbations. In the same manner as in the first order case, the next step is to use the "algebraic identity" (109) to eliminate ${}^{(2)}H_0^{\rm RW}{}_{,t}$ in (106) and (108), and thereby to reduce the system to a set of two coupled linear partial differential equations (with "sources") for ${}^{(2)}K^{\rm RW}$ and ${}^{(2)}H_1^{\rm RW}$. It is immediate to check that the part linear in ${}^{(2)}K^{\rm RW}{}_{,t}$ and ${}^{(2)}H_1^{\rm RW}$ has exactly the same form as the corresponding system for ${}^{(1)}K^{\rm RW}$ and ${}^{(1)}H_1^{\rm RW}$ in the first order perturbation case, as indeed it must. We may then, as in the first order perturbation treatment, introduce a "diagonalization" procedure by defining the functions ${}^{(2)}\chi$ and ${}^{(2)}\widehat{R}$, such that

$$\frac{\partial\ {}^{(2)}K^{\rm RW}}{\partial t} = f(r)\ {}^{(2)}\chi + g(r)\frac{\partial\ {}^{(2)}\widehat{R}}{\partial t}\quad ,\quad {}^{(2)}H_1^{\rm RW} = h(r)\ {}^{(2)}\chi + k(r)\ {}^{(2)}\widehat{R} \tag{110}$$

where f, g, h and k are the same as in (66). Using this transformation we may check that the system for ${}^{(2)}K^{\rm RW}{}_{,t}$ and ${}^{(2)}H_1^{\rm RW}$ is equivalent to

$$\frac{\partial\ {}^{(2)}\chi}{\partial r^*} = {}^{(2)}\widehat{R} + \widehat{\mathcal{S}}_{ZK}\ ,\quad \frac{\partial\ {}^{(2)}\widehat{R}}{\partial r^*} = \left[V(r^*) + \frac{\partial^2}{\partial t^2}\right]\ {}^{(2)}\chi + \widehat{\mathcal{S}}_{ZR}\ . \tag{111}$$

This, again, implies that ${}^{(2)}\chi$ satisfies an equation of the form,

$$\frac{\partial^2}{\partial r^{*2}}\ {}^{(2)}\chi - \frac{\partial^2}{\partial t^2}\ {}^{(2)}\chi - V(r^*)\ {}^{(2)}\chi + \mathcal{S}_Z = 0 \tag{112}$$

where $\mathcal{S}_Z$ is the "source" term for this second order Zerilli equation.

The procedure for specifying the Cauchy data for (112) is patterned on that for first order calculations. From our family of initial value solutions, we suppose that we have the 3-metric γ_{ij} and the extrinsic curvature K_{ij} to second order as well as first. We assume, for convenience, that the initial value solutions is in a spacetime gauge with ${}^{(2)}\widetilde{h}_0 = {}^{(2)}\widetilde{H}_1 = {}^{(2)}\widetilde{H}_0 = 0$. With (61) we then have the initial values of the perturbation functions ${}^{(2)}\widetilde{K}$,

${}^{(2)}\widetilde{G}$, ${}^{(2)}\widetilde{H}_2$, ${}^{(2)}\widetilde{h}_1$, and their first time derivatives. We next use the second order equivalent of (44) — (47). It should be recalled that these equations are the result of a purely second order gauge transformation; the second order version of (44) — (47) is precisely the same as the first order, with "1" superscripts replaced by "2". There are no additional terms quadratic in first order perturbations. These relations allow us to compute the initial values of ${}^{(2)}K^{(\rm RW)}$, ${}^{(2)}\dot{K}^{(\rm RW)}$ and ${}^{(2)}H_1^{(\rm RW)}$. The next step, following the first order pattern, is to use the second order Einstein equations

$$r\frac{\partial\ {}^{(2)}H_1^{RW}}{\partial t} = \frac{2M}{r}\ {}^{(2)}H_2^{RW} + (r-2M)\left(\frac{\partial\ {}^{(2)}H_2^{RW}}{\partial r} - \frac{\partial\ {}^{(2)}K^{RW}}{\partial r}\right) + \mathcal{S}_{H1t}^{\rm RW} \tag{113}$$

$$\frac{r^3}{r-2M}\frac{\partial^2\ {}^{(2)}K^{RW}}{\partial t^2} = r(r-2M)\frac{\partial^2\ {}^{(2)}K^{RW}}{\partial r^2} + 2(2r-3M)\frac{\partial\ {}^{(2)}K^{RW}}{\partial r}$$

$$-\,2(r-2M)\frac{\partial\ {}^{(2)}H_2^{RW}}{\partial r} + 2r\frac{\partial\ {}^{(2)}H_1^{RW}}{\partial t} + (\ell-1)(\ell+2)\ {}^{(2)}K^{RW} - 2\ {}^{(2)}H_2^{RW} + \mathcal{S}_{Ktt}^{\rm RW}, \tag{114}$$

the second order equivalents of (71) and (72), to find the initial values of ${}^{(2)}\dot{H}_1^{RW}$ and ${}^{(2)}\ddot{K}^{RW}$. With these and the definitions in (110) the determination of ${}^{(2)}\chi$ and ${}^{(2)}\dot{\chi}$, at $t=0$, is complete.

B. Radiation

Above we have given a description of how to solve for the second order Zerilli function ${}^{(2)}\chi$. It is important to understand that this second order perturbation cannot be considered as a second order correction to ${}^{(1)}\chi$. Rather, we must transform to coordinates which are asymptotically flat (AF) at least to second order in the perturbations. In this AF coordinates system, the dominant terms as $r\to\infty$ are the transverse K and G terms, which fall off as $1/r$. Most important, in the AF coordinate system the K and G terms give us the intensity of the outgoing radiation. The amplitude of the outgoing radiation, correct to first and second order in perturbation theory, therefore, is given by K and G correct to first and second order, in a gauge that is AF to first and second order.

In the problem of finding the radiation to second order there are calculational and conceptual problems that arise that are not present in the purely first order problem. In discussing these we start by pointing out how, in principle, the second order radiation could be computed in a straightforward way. We could, *ab initio*, invoke an AF gauge. (The specification of a unique AF gauge is not itself well defined, but that is not the issue here.) In this AF gauge we could write out the Einstein equations to first and to second order. The source terms ${}^{(2)}\mathcal{T}_{\alpha\beta}$ would have a form different from the form we use, based on a ${}^{(2)}\mathcal{T}_{\alpha\beta}$ computed in a RW gauge. And the various source terms $\mathcal{S}_K^{\rm AF}$, $\mathcal{S}_{H1}^{\rm AF}$, $\mathcal{S}_{H2}^{\rm AF}$, $\mathcal{S}_{AI}^{\rm AF}$, $\mathcal{S}_Z^{\rm AF}$ would have different values than the source terms $\mathcal{S}_K^{\rm RW}$, $\mathcal{S}_{H1}^{\rm RW}$, $\mathcal{S}_{H2}^{\rm RW}$, $\mathcal{S}_{AI}^{\rm RW}$, $\mathcal{S}_Z^{\rm RW}$. The resulting ${}^{(2)}\chi$ is a combination of second order RW perturbation functions, and as in the first order case, the relationship to K and G in a second order AF gauge can be found by a second order equivalent of a first order procedure. In practice one cannot use a first order AF gauge. The enormous complexity of the source term underscores the need for a simple, as well as definitive choice of gauge. Our approach is to work in a first order RW gauge and to compute $\mathcal{S}_K^{\rm RW}$, $\mathcal{S}_{H1}^{\rm RW}$, $\mathcal{S}_{H2}^{\rm RW}$, $\mathcal{S}_{AI}^{\rm RW}$, $\mathcal{S}_Z^{\rm RW}$.

We will then, in principle compute ${}^{(2)}\chi$ in a first order RW gauge. In practice, there is a new difficulty not present in the first order problem. When the first order RW gauge is used, the source term $\mathcal{S}_Z^{\rm RW}$ in the second order Zerilli equation diverges at large r, and any solution ${}^{(2)}\chi$ must diverge. This is of course a gauge effect (the effect of computing the sources in a first order RW gauge) and not an indication of a physical divergence. But as a practical matter we cannot do computations with a divergent quantity. In practice, therefore, we need to modify the problem to make it numerically tractable. This is possible since one does not spoil the invariance of ${}^{(2)}\chi$ under pure second order gauge transformations by adding pieces that are quadratic in the first order perturbations. This highlights the fact that ${}^{(2)}\chi$ is far from a unique quantity, in fact there is an infinite family of possible candidates for a "second order Zerilli function". We therefore add a known term to ${}^{(2)}\chi$ to cancel the divergent large-r behavior and to define a "renormalized" ${}^{(2)}\chi_n$ by

$${}^{(2)}\chi_n = {}^{(2)}\chi + \Gamma \tag{115}$$

which satisfies

$$\frac{\partial^2 \ {}^{(2)}\chi_n}{\partial r^{*2}} - \frac{\partial^2 \ {}^{(2)}\chi_n}{\partial t^2} - V(r^*) \ {}^{(2)}\chi_n = - \ {}^{(2)}\mathcal{S}_{Zn}$$
$$= -\frac{\partial^2 \Gamma}{\partial r^{*2}} + \frac{\partial^2 \Gamma}{\partial t^2} + V(r^*)\Gamma - \ {}^{(2)}\mathcal{S}_Z \tag{116}$$

Here Γ is a *known* expression quadratic in the (RW gauge) first order perturbations constructed so that the extra terms on the right hand side of (116) cancel the dominant large-r behavior. Since Γ is known, numerically solving for ${}^{(2)}\chi_n$ is equivalent to solving for ${}^{(2)}\chi$.

The explicit forms of source terms depend on the details of the problem and, in particular, on how the various first order multipoles contribute. We do not, therefore, give explicit general expressions for the source terms. $\mathcal{S}_K^{\rm RW}$, $\mathcal{S}_{H1}^{\rm RW}$, $\mathcal{S}_{H2}^{\rm RW}$, $\mathcal{S}_{AI}^{\rm RW}$, $\mathcal{S}_Z^{\rm RW}$, $\mathcal{S}_{Zn}^{\rm RW}$. Rather, as an example, we consider the case that contributions to the second order even parity quadrupole come exclusively from the first order even parity quadrupole. This specialization, in fact, applies to the particular configurations to which second order analysis has already been applied [18]– [20]. In this case, the renormalization is accomplished with

$$\chi_n^{(2)} = \chi^{(2)} - \frac{1}{7}\sqrt{\frac{\pi}{5}}\left[\frac{r^2}{2r+3M} \ {}^{(1)}K^{\rm RW} \ {}^{(1)}K^{\rm RW}{}_{,t} + (\ {}^{(1)}K^{\rm RW})^2\right] \tag{117}$$

and the resulting source $\mathcal{S}_{Zn}^{\rm RW}$, a quadratic expression in the first order perturbation, and some of their space and time derivatives, may be written entirely in terms of ${}^{(1)}\psi$, using (79) – (81). The explicit expression [1] is given in Eq. (18), in [19].

The process of extracting information about radiation from the solution for $\chi_n^{(2)}(t,r)$ follows the general pattern of the first order problem. One starts by writing the solution for ${}^{(2)}\chi_n(t,r)$ in the form

$${}^{(2)}\chi_n(t,r) = \ {}^{(2)}F_a(t-r^*) + \ {}^{(2)}F_b(t-r^*)/r + \ {}^{(2)}F_c(t-r^*)/r^2 + \mathcal{O}(1/r^3) \ . \tag{118}$$

From the second order Zerilli equation (112), one can show that ${}^{(2)}F_b$, ${}^{(2)}F_c$, etc. are determined once ${}^{(2)}F_a$ and the first order RW perturbations are known. For the special (but important) case that the only relevant first order perturbations are the even parity axisymmetric quadrupole perturbations, we have

$$\begin{aligned}{}^{(2)}F_b(t) = 3\int^t \ {}^{(2)}F_a dt' + \frac{2}{7}\sqrt{\frac{\pi}{5}}\Bigg\{ &\frac{1}{3}\frac{d \ {}^{(1)}F}{dt}\frac{d^4 \ {}^{(1)}F}{dt^4} + \frac{d^2 \ {}^{(1)}F}{dt^2}\frac{d^3 \ {}^{(1)}F}{dt^3} \\ &+\frac{1}{3}\int^t \left(\frac{d^3 \ {}^{(1)}F}{dt'^3}\right)^2 dt' + \frac{M}{6}\left[\frac{d^2 \ {}^{(1)}F}{dt^2}\frac{d^4 \ {}^{(1)}F}{dt^4} - \left(\frac{d^3 \ {}^{(1)}F}{dt^3}\right)\right]\Bigg\} .\end{aligned} \tag{119}$$

where ${}^{(1)}F$ is the asymptotic function introduced in (92), and we may obtain similar expressions for ${}^{(2)}F_c(t)$, and higher order coefficients.

With this form for $\chi^{(2)}(t,r)$, one then uses the second order system in (106)— (110), to solve for the second order RW functions. Those functions will, of course, diverge at large r due to now familiar gauge effects. This solution gives results, in terms of the asymptotic functions ${}^{(2)}F_a$, ${}^{(2)}F_b$, and ${}^{(2)}F_c$, for the functions ${}^{(2)}K^{\rm RW}$, ${}^{(2)}H_0^{\rm RW}$, ${}^{(2)}H_1^{\rm RW}$, ${}^{(2)}H_2^{\rm RW}$, which are perturbations in a gauge that is RW to both first and second order. To extract information about outgoing radiation we must now make two asymptotic gauge transformations. First we must transform from the first order RW gauge to an "intermediate" gauge, the result of a first order transformation that makes our coordinates system AF to first, but not to second, order. We have already discussed this transformation in Sec. IIIB, in particular in equations (95) — (101), and found the asymptotic form of the gauge transformation functions ${}^{(1)}\alpha_0^{\ell,m}$, ${}^{(1)}\alpha_1^{\ell,m}$, ${}^{(1)}\alpha_2^{\ell,m}$, for each multipole. In Sec. IIIB we were concerned only with the effect of this transformation on first order perturbations. Here we are concerned with the effect of the transformations on second order perturbations, and to second order, the effect of the transformation will be quadratic in the gauge functions ${}^{(1)}\alpha_0^{\ell,m}$, ${}^{(1)}\alpha_1^{\ell,m}$, ${}^{(1)}\alpha_2^{\ell,m}$, and will have the form, for example

$${}^{(2)}H_0^{\rm INT} = \ {}^{(2)}H_0^{\rm RW} + \mathrm{quad} \ , \tag{120}$$

[1] As correctly pointed out by Davies [29], there is a misprint in reference [19], the term involving $\dot{\psi}^2$ should be multiplied by $1/\mu^3$ (if not it would not be even dimensionally right). For the historical record, the misprint occurred in typing the manuscript, i.e. the correct formula was used in the numerical results derived with this formalism, e.g. in reference [18].

where "INT" represents the intermediate gauge, and "quad" stands for some expression quadratic in the first order terms, either in products of the ${}^{(1)}\alpha$ gauge functions, or products of ${}^{(1)}\alpha$ gauge functions with first order perturbations. For the first order effects all equations were linear, and each multipole of the ${}^{(1)}\alpha^{(\ell,m)}$ gauge functions had an effect only on the same multipole of the metric perturbations. In the second order effects, the multipoles mix, and no useful general expression can be given for "quad." In our most familiar case, when the only first order perturbations of concern are the even parity, axisymmetric, quadrupole perturbations, the explicit gauge transformation is, for example,

$$
\begin{aligned}
{}^{(2)}K^{\rm INT} = {}^{(2)}K^{\rm RW} - \frac{4}{7r}\sqrt{\frac{\pi}{5}}\Big\{\Big[&-r\,{}^{(1)}\alpha_0\,{}^{(1)}K^{\rm RW}{}_{,t} - {}^{(1)}\alpha_1\,{}^{(1)}K^{\rm RW}{}_{,r} \\
&+9r\,{}^{(1)}\alpha_2\,{}^{(1)}K^{\rm RW} - 2\,{}^{(1)}\alpha_1\,{}^{(1)}K^{\rm RW}\Big] + \frac{1}{r^2(r-2M)}\Big[-2Mr\,{}^{(1)}\alpha_1^2 - 6r^2\,{}^{(1)}\alpha_0^2 \\
&+7r^2\,{}^{(1)}\alpha_1^2 + 24r^4\,{}^{(1)}\alpha_2^2 - 24M^2\,{}^{(1)}\alpha_0^2 + 36Mr^2\,{}^{(1)}\alpha_1\,{}^{(1)}\alpha_2 - 18r^3\,{}^{(1)}\alpha_1\,{}^{(1)}\alpha_2 \\
&-48Mr^3\,{}^{(1)}\alpha_2^2 + 24Mr\,{}^{(1)}\alpha_0^2 - 4Mr^2\,{}^{(1)}\alpha_1\,{}^{(1)}\alpha_{1,r} - 4Mr^2\,{}^{(1)}\alpha_0\,{}^{(1)}\alpha_{1,t} \\
&+2r^3\,{}^{(1)}\alpha_1\,{}^{(1)}\alpha_{1,r} + 2r^3\,{}^{(1)}\alpha_0\,{}^{(1)}\alpha_{1,t}\Big]\Big\}\,. \qquad (121)
\end{aligned}
$$

With such expressions, and with the already established asymptotic forms of the gauge functions given in (99), or (101), we now have the relationship between the asymptotic form of ${}^{(2)}\chi$ and of the metric perturbations in the intermediate gauge. Equivalently, we have the asymptotic forms of ${}^{(2)}H_0^{\rm INT}$, ${}^{(2)}H_1^{\rm INT}$, ${}^{(2)}H_2^{\rm INT}$, ${}^{(2)}h_0^{\rm INT}$, ${}^{(2)}h_1^{\rm INT}$, ${}^{(2)}K^{\rm INT}$, ${}^{(2)}G^{\rm INT}$, in terms of ${}^{(2)}F_a(t-r*)$, and functions related to it.

The second step of our process is to perform a purely second order transformation to take us from the intermediate gauge to a gauge that is AF to both first and second order. We denote the gauge functions that implement this (even parity, quadrupolar, axisymmetric) transformation with ${}^{(2)}\alpha$, that is, our second order gauge transformation uses

$$
\begin{aligned}
\xi^{(2)0} = {}^{(2)}\alpha_0(t,r)Y_\ell^m \quad &, \quad \xi^{(2)1} = {}^{(2)}\alpha_1(t,r)Y_\ell^m \\
\xi^{(2)2} = {}^{(2)}\alpha_2(t,r)\frac{\partial}{\partial\theta}Y_\ell^m \quad &, \quad \xi^{(2)3} = {}^{(2)}\alpha_2(t,r)\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}Y_\ell^m \qquad (122)
\end{aligned}
$$

for $\ell, m = 2, 0$. These gauge functions, from RW to AF, have the asymptotic form

$$
\begin{aligned}
{}^{(2)}\alpha_0 &= r\,{}^{(2)}\alpha_{0a}(t-r^*) + {}^{(2)}\alpha_{0b}(t-r^*) + {}^{(2)}\alpha_{0c}(t-r^*)/r + \cdots \\
{}^{(2)}\alpha_1 &= r\,{}^{(2)}\alpha_{1a}(t-r^*) + {}^{(2)}\alpha_{1b}(t-r^*) + {}^{(2)}\alpha_{1c}(t-r^*)/r + \cdots \qquad (123) \\
{}^{(2)}\alpha_2 &= {}^{(2)}\alpha_{2a}(t-r^*)/r + {}^{(2)}\alpha_{2b}(t-r^*)/r^2 + {}^{(2)}\alpha_{2c}(t-r^*)/r^3 + \cdots\ .
\end{aligned}
$$

In the special case that the only first order perturbations that contribute to the second order quadrupole equations are the even parity axisymmetric quadrupole perturbations, the coefficient functions are given by

$$ {}^{(2)}\alpha_{0a}(t) = -\frac{1}{2}\,{}^{(2)}F_a(t) + \frac{1}{63}\sqrt{\frac{\pi}{5}}\left({}^{(1)}F'''(t)\right)^2 \qquad (124) $$

$$ {}^{(2)}\alpha_{1a}(t) = -\frac{1}{2}\,{}^{(2)}F_a(t) + \frac{1}{63}\sqrt{\frac{\pi}{5}}\left({}^{(1)}F'''(t)\right)^2 \qquad (125) $$

$$ {}^{(2)}\alpha_{2a}(t) = -\frac{1}{2}\int^t {}^{(2)}F_a(t')dt' + \frac{1}{126}\sqrt{\frac{\pi}{5}}\,{}^{(1)}F''(t)\,{}^{(1)}F'''(t) \qquad (126) $$

$$ {}^{(2)}\alpha_{0b}(t) = -\int^t {}^{(2)}F_a(t')dt' - M\,{}^{(2)}F_a(t) - \frac{M}{63}\sqrt{\frac{\pi}{5}}\left({}^{(1)}F'''(t)\right)^2 \qquad (127) $$

$$ {}^{(2)}\alpha_{1b}(t) = -\frac{3}{2}\int^t {}^{(2)}F_a(t')dt' \qquad (128) $$

$$ {}^{(2)}\alpha_{2b}(t) = -\int^t\left[\int^{t'} {}^{(2)}F_a(t'')dt''\right]dt' - \frac{2}{63}\sqrt{\frac{\pi}{5}}\,{}^{(1)}F'(t)\,{}^{(1)}F'''(t) + \frac{1}{126}\sqrt{\frac{\pi}{5}}\left({}^{(1)}F''(t)\right)^2 \qquad (129) $$

The function ${}^{(1)}F$ is known from the solution to the first order problem, and ${}^{(2)}F_a$ is defined in (118) as the asymptotic part of the second order Zerilli function.

The result of the two step gauge transformation is expressions, in terms of ${}^{(2)}F_a(t-r^*)$, and of first order functions, for the asymptotic second order metric perturbations in a (first and second order) AF gauge. Among these relations, we have ${}^{(2)}G^{\rm AF}$ and ${}^{(2)}K^{\rm AF}$ that carry information about the radiation. For the special case of only even parity axisymmetric quadrupole first order perturbations the results is

$$\frac{\partial\ {}^{(2)}G^{\rm AF}}{\partial t} = \frac{1}{r}\left\{\ {}^{(2)}F_a(t-r^*) + \frac{2}{63}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left[\ {}^{(1)}F''(t-r^*)\ {}^{(1)}F'''(t-r^*)\right]\right\} + O(r^{-2}) \tag{130}$$

$$\frac{\partial\ {}^{(2)}K^{\rm AF}}{\partial t} = \frac{3}{r}\left\{\ {}^{(2)}F_a(t-r^*) + \frac{2}{63}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left[\ {}^{(1)}F''(t-r^*)\ {}^{(1)}F'''(t-r^*)\right]\right\} + O(r^{-2})\ . \tag{131}$$

In terms of the actual results ${}^{(1)}\psi$ and ${}^{(2)}\chi_n$ of first and second order computations, these can be written.

$$\frac{\partial\ {}^{(2)}G^{\rm AF}}{\partial t} = \frac{1}{r}\left[\ {}^{(2)}\chi_n(t,r) + \frac{2}{7}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left(\ {}^{(1)}\psi(t,r)\frac{\partial\ {}^{(1)}\psi(t,r)}{\partial t}\right)\right] + O(r^{-2}) \tag{132}$$

$$\frac{\partial\ {}^{(2)}K^{\rm AF}}{\partial t} = \frac{3}{r}\left[\ {}^{(2)}\chi_n(t,r) + \frac{2}{7}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left(\ {}^{(1)}\psi(t,r)\frac{\partial\ {}^{(1)}\psi(t,r)}{\partial t}\right)\right] + O(r^{-2})\ . \tag{133}$$

All information about gravitational wave energy is carried by G and F in an AF gauge and, in (88), we have seen that ${}^{(1)}\dot{G}^{\rm AF} = {}^{(1)}\dot{K}^{\rm AF}/3 = {}^{(1)}\chi(t,r)/r$, for $\ell = 2$. We may therefore interpret the expression in brackets in ${}^{(2)}\dot{G}^{\rm AF}$ as the "second order correction to the gravitational wave amplitude χ". In particular, we have that the gravitational wave quadrupole power is

$$\text{Power} = \frac{6\pi}{25}\left\{\ {}^{(1)}\chi + \epsilon\left[\ {}^{(2)}\chi + \frac{2}{7}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left(\ {}^{(1)}\psi(t,r)\frac{\partial\ {}^{(1)}\psi(t,r)}{\partial t}\right)\right]\right\}^2\ . \tag{134}$$

We may choose to use the expression in (134), or to keep only the terms which are explicitly second order, and compute the energy from

$$\text{Power} = \frac{6\pi}{25}\left\{\ {}^{(1)}\chi + 2\epsilon\ {}^{(1)}\chi\left[\ {}^{(2)}\chi + \frac{2}{7}\sqrt{\frac{\pi}{5}}\frac{\partial}{\partial t}\left(\ {}^{(1)}\psi(t,r)\frac{\partial\ {}^{(1)}\psi(t,r)}{\partial t}\right)\right]\right\}\ . \tag{135}$$

which differs from (134) to third order. For comparison with numerical work [18,20] we take the expression in curly brackets in (134), aside from overall normalization, to be the gravitational wave amplitude correct to second order. We compare radiated energy to the time integral of the power given in (135), although the equally justifiable expression in (134) turned out to give better agreement with the numerical results used.

C. Second order techniques for collisions

The first and simplest application of second order calculations to collisions was the analysis of the collision starting with the Misner initial data described in (3) – (10). The family of spacetimes that evolves from these data is described by two parameters, but one of them is a trivial overall scaling (say the initial ADM mass). The remaining parameter μ_0 gives a dimensionless measure of initial separation and this parameter (more properly, some function of this parameter) is the basis of our ordering of perturbations. [See the discussion following (10).] The computations based on this scheme have been presented [18], and a comparison given of numerical relativity results, results of first order perturbation computations, and the results of perturbation theory to second order. The comparison showed precisely the pattern predicted: where the results of first order computations and of second oder computations began to diverge (which occurred for parameter μ_0 around 1.8) the results of either order started to diverge significantly from numerical results. This confirmed that second order perturbation would have told us the limiting range of perturbation results if we had not had available the results of numerical relativity.

Subsequent applications of perturbation theory have involved additional complications. Perturbation theory has been applied [30] to spacetimes evolving from Bowen-York [31] initial data corresponding to two equal mass holes which are initially moving symmetrically toward each other. In this case we have three parameters. One is an overall scaling and can be taken to be the initial ADM mass M of the spacetime. A second parameter is the magnitude of the initial momentum P of each hole, and a third is some measure L of the initial distance between the holes.

Aside from the overall scaling, the family of spacetimes can be characterized by two dimensionless parameters, say P/M and L/M. The perturbation analysis reported in [30] treated both of these parameters as small. In order to apply perturbation methods to such a multiparameter family, it is useful to consider a curve through the parameter space. Such a curve gives us a one parameter family of spacetimes, and we can then apply standard methods. The curve through the parameter space, however, is not unique. As an example, let us suppose that we are considering initial data for $P/M = 0.3$ and $L/M = 0.1$. To find perturbation results for this example, we could consider that we are on a curve $P/M = 3L/M$, and L/M is our perturbation parameter. But we could equally well treat the spacetime as a point along the curve $P/M = 30(L/M)^2$. The perturbation analysis for close/slow (i.e., small P, small L) perturbation theory will depend on which curve in parameter space was chosen. The agreement with numerical relativity results will not be equally good for the two choices.

A different sort of choice was described following (28). It was pointed out that for second order perturbation theory, one could "feed back" all information about the first order perturbations; this gives a result that differs only to third and higher order from "standard" second order perturbation theory. It was also pointed out that procedure destroys the spherical symmetry of the differential operators in the perturbation equations and enormously complicates the analysis.

There is, however, a way of using updating without paying the price of loss of symmetry, and it turns out to give an important improvement. We can update only the information about the monopole, thereby preserving spherical symmetry of the operators. This turns out to be quite important in the slow/close perturbation analysis of initially boosted holes [30,20]. The ADM mass of the spacetime turns out to be much more sensitive to the initial momentum, than the quadrupole deformation is. A first order change in the monopole mixes with a first order change in the quadrupole to produce a second order quadrupole deformation, so our second order radiation computations are influenced by the change in the mass. If we use the standard approach, the rapid growth of the ADM mass with momentum is the limiting factor that determines the disappointingly small range of applicability of perturbation theory. To avoid this limitation we can use a higher order estimate of the ADM mass. We have, in fact, used a numerical (rather than perturbative) computation of the ADM mass corresponding to a particular set of parameters P and L. This procedure results in an enormous improvement in the range over which perturbation calculations can be used for radiation. The reason for this is clear in the following: For values of P large enough so that the ADM mass increases by several hundred percent, the radiated energy is still very small.

This same phenomenon was found in our analysis of the radiation generated as a single spinning Bowen-York [31] hole evolves to its final Kerr state. This problem was analyzed in the limit of slow rotation, so that it could be treated as a perturbation of a Schwarzschild hole [16]. The effect of the spin on the ADM mass was again found to be much larger than the effect on radiatable multipoles. Again, a numerical evaluation of the ADM mass was used to extend the range of the analysis.

V. SUMMARY

Buried, not too deeply we hope, in the equations of the previous section are a few general lessons that justify emphasis. We see, for one thing, that in second order work, one relinquishes all the simplicity of linearity that characterizes first order perturbation calculations. Gauge transformations, in particular, have a completely different character when one works at second order; gauge transformations of first order change second order perturbations. In our approach to second order perturbations, these gauge transformations could not be avoided for a rather basic reason: One does the mathematics in one gauge, and the physics in others. Actual numerical second order computations have to be carried out in a gauge which is both convenient (i.e., minimizes the proliferation of terms) and (unlike the "asymptotically flat gauge") is definitive. In our approach that means the use of a system that satisfies the Regge–Wheeler gauge conditions to both first and second order. But the Regge–Wheeler gauge is generally not related to two other gauges that are of importance to the problem of evolving initial data and finding radiation. First, initial data will be found for the problem in some "initial data" gauge, and must be transformed to Regge–Wheeler gauge to give us Cauchy data for evolution. Secondly, the result of second order evolution must be transformed, asymptotically, to an asymptotically flat gauge in order for information to be extracted about outgoing wave amplitudes. One must explicitly perform gauge transformations in the process of making these calculations. And the gauge transformations require two steps. One must, in general, perform a first order gauge transformation carried out to second order, and next an exclusively second order gauge transformation.

There are, in principle, other ways to proceed with second order calculations. It is possible, in principle, to construct expressions which are formally gauge invariant to both first and second order. The same formal expressions could be used for setting the Cauchy data, for evolution, and for interpretation of the results. Such gauge invariant expressions would, in effect, have built into them the "gauge invariant" (i.e., unique) character of the Regge–Wheeler gauge. Since

the procedure for going from an arbitrary gauge to a (first and second order) Regge–Wheeler gauge is unambiguous and local, all second order Regge–Wheeler gauge quantities can be written as combinations of perturbations in a general gauge. The development of such a formalism is underway [32]. By their gauge invariant character such expressions would have built into them the gauge transformations, that we perform, in a manner of speaking, as an "external" process. The gauge invariant expressions will therefore be significantly more complex in appearance, but the important issue is not so much appearance, as suitability for computation. Since these computations (e.g., the evaluation of source terms) will be carried out numerically, a gauge invariant expression may turn out be more subject to round off error or less, and roundoff error can be a significant problem in expressions combining many high derivative terms.

In closing, we note that second order perturbation theory has turned out to be a great deal more difficult than linearized theory, but overcoming these difficulties is motivated by the fact that second order calculations are a great deal easier than numerical relativity. What can be (speculatively) said of third and higher order calculations? On the one hand we suggest that the step from second order to third and higher order might not be as painful as the step from first to second order. The step up to second order required developing the new tools for dealing with nonlinearities. With the pattern of those tools established, and with the conceptual issues faced, the next step up adds complexity, but, we believe, no new conceptual difficulties. The complexity added by each step up in order is considerable, but such complexity is not a crucial obstacle if the work is being done, as it certainly must be, by computers.

To balance this argument. that one should not be terrified of yet higher order perturbation calculations, it should be asked what is to be gained by such higher order results. The overwhelming motivation for second order work was the need to establish the range of validity of perturbation analysis. Third order calculations add nothing to this, so they are motivated only by the possibility of higher accuracy. But higher accuracy is guaranteed only if the perturbation series is convergent. There is no reason to expect that for large values of the expansion parameter the series is convergent, and for small values the marginal increase in accuracy would not seem to justify the work of going to higher order computations.

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